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# Vertex stabilizers of graphs and tracks, I ☆

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**Abstract**

This paper is devoted to the conjecture saying that, for any connected locally finite graph $\Gamma$ and any vertex-transitive group $G$ of automorphisms of $\Gamma$, at least one of the following assertions holds: (1) There exists an imprimitivity system $\sigma$ of $G$ on $V(\Gamma)$ with finite (maybe one-element) blocks such that the stabilizer of a vertex of the factor graph $\Gamma/\sigma$ in the induced group of automorphisms $G^\sigma$ is finite. (2) The graph $\Gamma$ is hyperbolic (i.e., for some positive integer $n$, the graph $\Gamma^n$ defined by $V(\Gamma^n) = V(\Gamma)$ and $E(\Gamma^n) = \{\{x, y\} : 0 < d_\Gamma(x, y) \leq n\}$ contains the regular tree of valency 3). Our approach to the conjecture consists in fixing a finite permutation group $R$ and considering the conjecture under the assumption that the stabilizer of a vertex of $\Gamma$ in $G$ induces on the neighborhood of the vertex a group permutation isomorphic to $R$. In the paper we elaborate a method (the modified track method) which allows us to prove the conjecture for many groups $R$. The paper consists of two parts. The present first part of the paper involves results on which the modified track method arguments are based, and a few first applications of the method. The second part is devoted to applications of the modified track method.



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## 1. Introduction

In this paper we deal with the following problem (for different $R$).

$(*)$ Fix a finite permutation group $R$. Let $\Gamma$ be a connected locally finite graph and $G$ a vertex-transitive group of automorphisms of $\Gamma$ such that the stabilizer of a vertex of $\Gamma$ in $G$ induces on the neighborhood of the vertex a group permutation isomorphic to $R$. Is it true that at least one of the following two assertions holds?

(1) There exists an imprimitivity system $\sigma$ of $G$ on $V(\Gamma)$ with finite (maybe one-element) blocks such that the stabilizer of a vertex of the factor graph $\Gamma/\sigma$ in the induced group of automorphisms $G^\sigma$ is finite.

(2) The graph $\Gamma$ is hyperbolic (i.e., for some positive integer $n$, the graph $\Gamma^n$ defined by $V(\Gamma^n) = V(\Gamma)$ and $E(\Gamma^n) = \{\{x, y\} : 0 < d_\Gamma(x, y) \leq n\}$ contains the regular tree of valency 3).

There are groups $R$ for which the problem $(*)$ is easy, but definitely there are many others $R$ for which this is not the case. The purpose of the present paper is to elaborate a method (the modified track method) allowing one to answer in the affirmative the problem $(*)$ in many such more complicated cases of groups $R$.

To put the discussion in proper perspective, it will be useful to look at the problem $(*)$ from two points of view.

On the one hand, in $(*)$ we can omit mentioning $R$ completely. As a result we get a conjecture which, in the present paper, is called the Main Conjecture. (This conjecture was formulated for the first time in [5] (see Conjecture $(**)$ on p. 120 of [5]). It was also formulated as Problem 12.87 in [4].) Now the problem $(*)$ for various $R$ turns into a case-by-case consideration of the Main Conjecture. This *local approach* to the Main Conjecture, consisting in choosing a finite permutation group $R$ and considering the Main Conjecture under the additional assumption that the stabilizer of a vertex of $\Gamma$ in $G$ induces on the neighborhood of the vertex a group coinciding with $R$ (i.e., consisting in considering the problem $(*)$ for various $R$), seems natural and productive since, for certain groups $R$, the corresponding result is of independent interest. In this paper, we prefer to call the problem $(*)$ for a group $R$ the Main Conjecture for the group $R$ or the Main Conjecture in the case of the group $R$.

On the other hand, one has the following Finite Vertex Stabilizer Reconstruction Problem (the FVSRP for short):

Fix a finite permutation group $R$. Let $\Gamma$ be a connected locally finite graph and $G$ a vertex-transitive group of automorphisms of $\Gamma$ such that the stabilizer of a vertex of $\Gamma$ in $G$ is finite and induces on the neighborhood of the vertex a group permutation isomorphic to $R$. What is the possible structure of the stabilizer of a vertex of $\Gamma$ in $G$ in that case?

Like $(*)$, the FVSRP is interesting for many, but not for all groups $R$. Originated in [11] and [12], the FVSRP (for certain groups $R$) is closely connected to the local analysis in the finite group theory. We briefly consider the FVSRP in that context in the Appendix (see the second part of the present paper). In the Appendix we also outline the track method for the FVSRP which allowed recent progress in the FVSRP (see [10]). The reason is that some arguments of the track method for the FVSRP can be modified to be applied to the problem $(*)$ for many interesting $R$. Actually it is a purpose of the present paper to elaborate such a modified track method to answer in the affirmative the problem $(*)$ in many interesting cases of groups $R$. But the modified track method is not the only link between the problem $(*)$ and the FVSRP. The problem $(*)$ leads, in a natural way, to the FVSRP as soon as we are interested in vertex stabilizers in the case when the assertion (1) from $(*)$ holds. We may look at this from the following point of view. In principle, in the FVSRP the condition that the stabilizer of a vertex of $\Gamma$ in $G$ is finite can be omitted. As a result the FVSRP turns into the G(eneral)VSRP. But such generalization seems too sweeping to hope to get a kind of description of the possible structure of the stabilizer of a vertex of $\Gamma$ in $G$ already for some concrete groups $R$. However, the assumption that the stabilizer of a vertex of $\Gamma$ in $G$ is infinite seems strong enough to restrict remarkably the structure of $\Gamma$ as formulated in $(*)$. From this point of view, $(*)$ reduces the GVSRP to the FVSRP "modulo" hyperbolic graphs.

The structure of the paper is as follows. In Section 2 we formulate the Main Conjecture explicitly and list a few known results which establish the Main Conjecture in some cases or can be used with that end. Some of these results play an important role in the subsequent sections. In addition, in this section we prove a fairly general result that is also useful in the following work, concerning composition factors of infinite vertex stabilizers (see Proposition 2.5 and Corollary 2.7). In Section 3 we begin to realize the local approach to the Main Conjecture, but in a rather straightforward way. As a result, in this section the Main Conjecture is proved in some "easy" cases. More complicated cases need however more delicate techniques. Such an advanced method (the modified track method) is developed starting with Section 4. In Section 4 some general results concerning tracks of graphs are proved. These results form a basis of the modified track method arguments. At the same time these results are of independent interest. In this connection, in this section our consideration is slightly more general than is needed in the subsequent sections. At the beginning of Section 5 we outline a framework of the modified track method. After that we derive some consequences of results of Section 4 adapted to using in the modified track method arguments. Finally, in this section we give first applications of the method within the local approach to the Main Conjecture (completing, in particular, the case when $R$ is a doubly transitive group with a simple socle different from $PSL_n(q)$, $n > 2$). More applications of the modified track method within the local approach to the Main Conjecture are given in the second part of the paper. The second part of the paper is concluded with the Appendix containing an expository account of the FVSRP with an emphasis on the track method.

There are many other—not considered in the present paper—applications of the modified track method within the local approach to the Main Conjecture. The purpose of the present paper is not to prove the Main Conjecture in as many cases as possible, but rather to elaborate and demonstrate with some interesting examples a new fairly general method.

Most of our notation and terminology is standard.

All graphs in this paper are undirected graphs without loops or multiple edges. Let $\Gamma$ be a graph. Then $V(\Gamma)$ is the vertex set of $\Gamma$ and $E(\Gamma)$ is the edge set of $\Gamma$. For $x \in V(\Gamma)$, $\Gamma(x)$ is the set of vertices of $\Gamma$ adjacent to $x$ in $\Gamma$ (the neighborhood of $x$ in $\Gamma$). The graph $\Gamma$ is *locally finite* if the valency $|\Gamma(x)|$ of any vertex $x$ of $\Gamma$ is finite. For a non-negative integer $s$, a *path* (*of length* $s$) of $\Gamma$ is a sequence $(x_0, \dots, x_s)$ of vertices of $\Gamma$ such that $\{x_i, x_{i+1}\} \in E(\Gamma)$ for all $0 \le i < s$. In the case $x_{i-1} \ne x_{i+1}$ for all $0 < i < s$, the path $(x_0, \dots, x_s)$ is called an *s-arc* of $\Gamma$. For $x, y \in V(\Gamma)$, $d_\Gamma(x, y)$ is the length of a shortest path of $\Gamma$ between $x$ and $y$ in the case where such a path exists, and $\infty$ otherwise. Thus $\Gamma$ is connected if and only if $d_\Gamma(x, y) < \infty$ for all $x, y \in V(\Gamma)$. For any non-negative integer $i$, the graph $\Gamma^i$ is defined by $V(\Gamma^i) = V(\Gamma)$ and $E(\Gamma^i) = \{\{x, y\} : 0 < d_\Gamma(x, y) \le i\}$. Thus, for any $x \in V(\Gamma)$ and any non-negative integer $i$, $\Gamma^i(x) = \{y \in V(\Gamma) : 0 < d_\Gamma(x, y) \le i\}$.

As usual, a graph $\Delta$ is a *subgraph* of $\Gamma$ (as before we assume that $\Gamma$ is an arbitrary graph) if $V(\Delta) \subseteq V(\Gamma)$ and $E(\Delta) \subseteq E(\Gamma)$. For $\emptyset \ne X \subseteq V(\Gamma)$, the *subgraph of $\Gamma$ generated by* $X$ is the graph with the vertex set $X$ and with the edge set $\{\{x', x''\} \in E(\Gamma) : \{x', x''\} \subseteq X\}$. A *connected component* of $\Gamma$ is a maximal subset of $V(\Gamma)$ generating a connected subgraph of $\Gamma$. A graph $\Delta$ is *contained in* $\Gamma$ if $\Delta$ is isomorphic to a subgraph of $\Gamma$. The graph $\Gamma$ is *hyperbolic* if, for some positive integer $n$, the graph $\Gamma^n$ contains the regular tree of valency 3. (Observe that, if $\Gamma$ is hyperbolic, then for any positive integer $d$ there exists a positive integer $n$ such that $\Gamma^n$ contains the regular tree of valency $d$.)

As usual, if $\Gamma$ is a graph and $\sigma$ is a partition of $V(\Gamma)$, the *factor graph* $\Gamma/\sigma$ is the graph whose vertex set is the set of elements of $\sigma$ and whose edge set consists of all pairs of different elements of $\sigma$ such that some vertex of one element of the pair is adjacent to some vertex of another element of the pair.

Let $G$ be a permutation group on a set $X$. Then for $x \in X$ and $X' \subseteq X$, $G_x$ denotes the stabilizer of $x$ in $G$, $G_{X'}$ denotes the pointwise stabilizer of the set $X'$ in $G$, and $G_{\{X'\}}$ denotes the (global) stabilizer of the set $X'$ in $G$. For $x', x'', \dots \in X$, we also write $G_{x',x'',\dots}$ for the pointwise stabilizer of the set $\{x', x'', \dots\}$ in $G$. For $x \in X$, $G(x)$ denotes the $G$-orbit containing $x$. For a $G$-invariant subset $X'$ of $X$, $G^{X'}$ denotes the group induced by $G$ on $X'$. If $\sigma$ is a partition of $X$ and $x \in X$, then the element of $\sigma$ containing $x$ is denoted by $x^\sigma$. A partition $\sigma$ of $X$ is *$G$-invariant* if $g(Y) \in \sigma$ for any $g \in G$ and any $Y \in \sigma$. If $\sigma$ is a $G$-invariant partition, then the group induced by $G$ on $\sigma$ is denoted by $G^\sigma$. In the case when $G$ is transitive, $G$-invariant partitions are called imprimitivity systems of $G$. (Thus we regard the partitions of $X$ consisting of $X$ or of singletons as (trivial) imprimitivity systems of the transitive group $G$.) Elements of imprimitivity systems of $G$ are called blocks of imprimitivity of $G$.

Recall that a permutation group $G$ on a set $X$ is *quasiprimitive* if $G$ and all non-trivial normal subgroups of $G$ are transitive (on $X$). Of course, if $G$ is primitive, then $G$ is quasiprimitive.

Let $\Gamma$ be a graph and $G \le Aut(\Gamma)$. (In the present paper, a group of automorphisms of a graph is regarded as a permutation group on the vertex set of the graph.) For a non-negative integer $i$ and $x \in V(\Gamma)$, $G_x^{[i]}$ denotes the pointwise stabilizer of $\Gamma^i(x) \cup \{x\}$ in $G$. (In particular, $G_x^{[0]} = G_x$.) More generally, for $x', x'', \dots \in V(\Gamma)$, define $G_{x',x'',\dots}^{[i]}$ to be the pointwise stabilizer in $G$ of the set $\Gamma^i(x') \cup \{x'\} \cup \Gamma^i(x'') \cup \{x''\} \cup \dots$. It is easy to see that in the case where $\Gamma$ is connected, either each or none of the $G$-orbits on $V(\Gamma)$ generates a connected subgraph in the graph $\Gamma^d$ for some, depending on the $G$-orbit, positive integer $d$. If $\Gamma$ is connected and locally finite, then obviously either each or none of the $G$-orbits on $V(\Gamma)$ is finite. If $\sigma$ is a $G$-invariant partition of $V(\Gamma)$ (in particular, if $\sigma$ is an imprimitivity system of $G$), then obviously $G^\sigma \le Aut(\Gamma/\sigma)$.

Let $X$ be a set. $\mathbf{Z}$-indexed sequences $\ldots, x'_{-1}, x'_0, x'_1, \ldots$ and $\ldots, x''_{-1}, x''_0, x''_1, \ldots$ of elements of $X$ are *shift equivalent* if there exists $s \in \mathbf{Z}$ such that $x''_i = x'_{i+s}$ for all $i \in \mathbf{Z}$. The classes of shift equivalent $\mathbf{Z}$-indexed sequences of elements of $X$ will be called *two-way infinite sequences* of elements of $X$. For a $\mathbf{Z}$-indexed sequence $\ldots, x_{-1}, x_0, x_1, \ldots$ of elements of $X$, the corresponding two-way infinite sequence will be denoted by $(\ldots, x_{-1}, x_0, x_1, \ldots)$. For a two-way infinite sequence $T = (\ldots, x_{-1}, x_0, x_1, \ldots)$ of elements of $X$, put $T^{-1} := (\ldots, x_1, x_0, x_{-1}, \ldots)$.

Let $\Gamma$ be a graph, $g \in Aut(\Gamma)$ and $G \leq Aut(\Gamma)$. A *generalized g-track* of $\Gamma$ is a two-way infinite sequence $(\ldots, x_{-1}, x_0, x_1, \ldots)$ of vertices of $\Gamma$ such that $x_{i+1} = g(x_i)$ for all $i \in \mathbf{Z}$. A *generalized G-track* of $\Gamma$ is a generalized $g'$-track of $\Gamma$ for some $g' \in G$. Let $T = (\ldots, x_{-1}, x_0, x_1, \ldots)$ be a generalized $g$-track of $\Gamma$ or a generalized $G$-track of $\Gamma$. Then $T$ is *infinite* (respectively *finite*) if the subset $\{x_i : i \in \mathbf{Z}\}$ of $V(\Gamma)$ is infinite (respectively finite). $T$ is *trivial* if all vertices $x_i$, $i \in \mathbf{Z}$, coincide. For $z \in V(\Gamma)$, $T$ *passes through* $z$, if $z = x_i$ for some $i \in \mathbf{Z}$. If $T = (\ldots, x_{-1}, x_0, x_1, \ldots)$ is a generalized $g$-track of $\Gamma$ (respectively a generalized $G$-track of $\Gamma$) such that $\{x_i, x_{i+1}\} \in E(\Gamma)$ for all (equivalently, for some) $i \in \mathbf{Z}$, then $T$ is called a *g-track* of $\Gamma$ (respectively a *G-track* of $\Gamma$).

## 2. The Main Conjecture and some related results

The following conjecture is the main subject under consideration in the present paper.

**Main Conjecture.** *Let $\Gamma$ be a connected locally finite graph, and let $G$ be a vertex-transitive group of automorphisms of $\Gamma$. Then at least one of the following two assertions holds:*

(1) *There exists an imprimitivity system $\sigma$ of $G$ on $V(\Gamma)$ with finite (maybe one-element) blocks such that, for $x \in V(\Gamma)$, the stabilizer $(G^\sigma)_{x^\sigma}$ of the vertex $x^\sigma$ in $G^\sigma$ is finite.*
(2) *The graph $\Gamma$ is hyperbolic.*

**Remark 2.1.** In connection with the Main Conjecture, the following terminology seems appropriate. Let $G$ be a vertex-transitive group of automorphisms of a connected locally finite graph $\Gamma$. Then $G$ has *essentially infinite* vertex stabilizers if the possibility (1) from the Main Conjecture does not hold for $\Gamma$ and $G$. Now the Main Conjecture says that any connected locally finite graph admitting a vertex-transitive group of automorphisms with essentially infinite vertex stabilizers is hyperbolic.

**Remark 2.2.** It is easy to see that to prove the Main Conjecture it is sufficient to consider the case $G = Aut(\Gamma)$. Observe also that, for $\Gamma$ and $G$ satisfying the hypothesis of the Main Conjecture and for $x \in V(\Gamma)$, if the group $G_x$ is infinite and the group $G_x^{\Gamma(x)}$ is quasiprimitive, then $G_x$ is essentially infinite. In fact, assume the group $G_x$ is infinite and the group $G_x^{\Gamma(x)}$ is quasiprimitive but the possibility (1) from the Main Conjecture holds for $\Gamma$ and $G$. Then, denoting by $K$ the kernel of the action of $G$ on $\sigma$, we have $K_x \neq 1$ and hence $K_x^{\Gamma(x)} \neq 1$. By quasiprimitivity of $G_x^{\Gamma(x)}$ this gives that $\Gamma(x)$ is contained in a block of $\sigma$. Since blocks of $\sigma$ are finite, it follows that $V(\Gamma)$ is finite, contradicting the assumption that $G_x$ is infinite.

**Remark 2.3.** The author would be rather surprised if the Main Conjecture were proved in general. At the same time the author is optimistic as regards the case where the group $G_x^{\Gamma(x)}$ is primitive.

As it has been said in the Introduction, the local approach to the Main Conjecture discussed in this paper consists in fixing a finite permutation group $R$ and considering the Main Conjecture under the additional assumption that $G_x^{\Gamma(x)}$ coincides with $R$ (as a permutation group). This approach is realized starting with Section 3. In the remainder of this section, we formulate a few known results which establish the validity of the Main Conjecture in some cases or can be used for this. Some of these results are used in the subsequent sections of the present paper. In addition, we prove a fairly general result which is also useful in the following work concerning composition factors of groups $G_x^{[n]}/G_x^{[n+1]}$ in the case when $G_x^{\Gamma(x)}$ is quasiprimitive and $G_x$ is infinite (see Proposition 2.5 and Corollary 2.7).

In the remainder of this section, $\Gamma$ and $G$ satisfy the hypothesis of the Main Conjecture, $x \in V(\Gamma)$ and $y \in \Gamma(x)$.

First we recall two concepts (of bounded automorphisms of graphs and of modular functions of groups of automorphisms of graphs) which are used below.

An automorphism $h$ of a connected graph $\Delta$ is called *bounded* if there exists a positive integer $c$ such that $d_\Delta(z, h(z)) \leq c$ for all $z \in V(\Delta)$. The set of all bounded automorphisms of $\Delta$ is a normal subgroup of $Aut(\Delta)$ denoted by $Aut_0(\Delta)$. By [6, Corollary 1], in the case where $\Delta$ is locally finite and $Aut_0(\Delta)$ is vertex-transitive, there is an imprimitivity system $\tau$ of $Aut(\Delta)$ on $V(\Delta)$ with finite blocks such that $(Aut_0(\Delta))^\tau \simeq \mathbf{Z}^d$ for some non-negative integer $d$. (It easily follows from this result that the blocks of $\tau$ are orbits of the subgroup of $Aut(\Delta)$ consisting of all bounded automorphisms of finite order of $\Delta$.)

If $\sigma$ is an imprimitivity system of $G$ on $V(\Gamma)$ with finite blocks, then the kernel of the natural homomorphism $G \to G^\sigma$ consists, obviously, of bounded automorphisms (of finite order) of $\Gamma$. It follows that in the case when $G_x$ is infinite but $(G^\sigma)_{x^\sigma}$ is finite, $G_x$ contains an infinite subgroup consisting of bounded automorphisms.

A connected graph $\Delta$ is *reduced* if any non-trivial bounded automorphism of $\Delta$ stabilizes no vertices of $\Delta$. By [8, Proposition 2.3], for any connected locally finite graph $\Delta$ admitting a vertex-transitive group of automorphisms, there exists an imprimitivity system $\tau$ of $Aut(\Delta)$ on $V(\Delta)$ with finite blocks (which may always be taken to be orbits of a normal subgroup of $Aut(\Gamma)$) such that $\Delta/\tau$ is reduced. Thus to prove the Main Conjecture it is sufficient to consider only reduced graphs $\Gamma$. Note that for reduced graphs $\Gamma$, the Main Conjecture says that $\Gamma$ is hyperbolic if $G_x$ is infinite.

Another concept has a topological flavor. The group $Aut(\Gamma)$ equipped with the topology of pointwise convergence is a locally compact group, and the stabilizer of $x$ in $Aut(\Gamma)$ is a compact open subgroup of $Aut(\Gamma)$ (see [7]). Let $\overline{G}$ be the closure of $G$ in $Aut(\Gamma)$. Then, for $z \in V(\Gamma)$ and for a finite subset $Z$ of $V(\Gamma)$, $Z$ is $G_z$-invariant if and only if $Z$ is $\overline{G}_z$-invariant. Moreover, in this case $G_z^Z = \overline{G}_z^Z$. In particular, we have $G_x^{\Gamma(x)} = \overline{G}_x^{\Gamma(x)}$. Thus to prove that the Main Conjecture is valid for any $\Gamma$ and $G$ with a fixed group $G_x^{\Gamma(x)} = R$ it is sufficient to consider only the case when $G$ is a closed subgroup of $Aut(\Gamma)$.

Suppose that $G$ is a closed subgroup of $Aut(\Gamma)$. (Thus $G$ is locally compact and $G_x$ is compact.) Then the modular function $\mathrm{Mod}_G$ of the topological group $G$ is defined.

In general, we define $\mathrm{Mod}_G$ as the restriction to $G$ of the function $\mathrm{Mod}_{\overline{G}}$ where $\overline{G}$ is the closure of $G$ in $Aut(\Gamma)$. $\mathrm{Mod}_G$ is a homomorphism of $G$ into the multiplicative group of positive real numbers which can be defined by

$$\mathrm{Mod}_G(g) = \frac{|G_x : G_{x,g(x)}|}{|G_{g(x)} : G_{x,g(x)}|} = \frac{|G_x(g(x))|}{|G_{g(x)}(x)|} = \frac{|G_x(g(x))|}{|G_x(g^{-1}(x))|}$$

for any $g \in G$. In the case where $\mathrm{Mod}_G(g) = 1$ for all $g \in G$, the group $G$ is called *unimodular*. Since $\overline{G} = \langle \overline{G}_x, G\rangle$ and $\mathrm{Mod}_{\overline{G}}(g) = 1$ for any $g \in \overline{G}_x$, $G$ is unimodular if and only if $\overline{G}$ is unimodular. Observe that unimodularity of $G$ is equivalent to each of the following conditions:

(i) $|G_{x'} : G_{x',y'}| = |G_{y'} : G_{x',y'}|$ for all (or some) $x' \in V(\Gamma)$ and all $y' \in V(\Gamma) \setminus \{x'\}$.
(ii) $|G_{x'} : G_{x',y'}| = |G_{y'} : G_{x',y'}|$ for all (or some) $x' \in V(\Gamma)$ and all $y' \in \Gamma(x')$.
(iii) $|G_{x',y'}^{\Gamma^i(x')}| = |G_{x',y'}^{\Gamma^i(y')}|$ for all (or some) $x' \in V(\Gamma)$, all $y' \in V(\Gamma) \setminus \{x'\}$ and all (or some) $i \geq d_\Gamma(x', y')$.
(iv) $|G_{x',y'}^{\Gamma(x')}| = |G_{x',y'}^{\Gamma(y')}|$ for all (or some) $x' \in V(\Gamma)$ and all $y' \in \Gamma(x')$.

In addition, if $G$ is unimodular and $(\ldots, x_{-1}, x_0, x_1, \ldots)$ is a generalized $g$-track of $\Gamma$, where $g \in G$, then

$$\begin{aligned} |(G^{[1]}_{x_i,\ldots,x_{i+k}})^{\Gamma(x_{i-1})}| &= |G^{[1]}_{x_i,\ldots,x_{i+k}} : G^{[1]}_{x_{i-1},x_i,\ldots,x_{i+k}}| \\ &= |G^{[1]}_{x_i,\ldots,x_{i+k}} : gG^{[1]}_{x_{i-1},x_i,\ldots,x_{i+k}}g^{-1}| \\ &= |G^{[1]}_{x_i,\ldots,x_{i+k}} : G^{[1]}_{x_i,\ldots,x_{i+k},x_{i+k+1}}| = |(G^{[1]}_{x_i,\ldots,x_{i+k}})^{\Gamma(x_{i+k+1})}| \end{aligned}$$

for any positive integer $k$. In particular,

$$|(G^{[1]}_{x_i,\ldots,x_{i+k}})^{\Gamma(x_{i-1})}| = |(G^{[1]}_{x_i,\ldots,x_{i+k}})^{\Gamma(x_{i+k+1})}| > 1 \tag{2.1}$$

in the case when $G^{[1]}_{x_1,\ldots,x_k} \neq G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ for a positive integer $k$.

In the subsequent sections of the present paper we need the following result (see [7, Theorem 2]):

(R1) *If $\Gamma$ is non-hyperbolic, then $G$ is unimodular.*

Now we formulate a few results from [5] related to the Main Conjecture. (Note that in [5] a somewhat different terminology is used.)

(R2) *The Main Conjecture is valid for $\Gamma$ and $G$ in the case where $\Gamma$ is a graph of valency* 3 (see [5, Theorem 3.1]).

(R3) *The possibility* (1) *from the Main Conjecture holds for $\Gamma$ and $G$ whenever this possibility holds for $\Gamma$ and $H$ where $H$ is a vertex-transitive normal subgroup of $G$* (*see* [5, Theorem 4.1]). *Thus the Main Conjecture is valid for $\Gamma$ and $G$ whenever the Main Conjecture is valid for $\Gamma$ and some vertex-transitive normal subgroup $H$ of $G$.*

(R4) *If $\Gamma$ is non-hyperbolic and $N$ is a normal subgroup of $G$ such that $G$ induces on the set of $N$-orbits on $V(\Gamma)$ a cyclic group, then there exists a positive integer $d$ such that each $N$-orbit on $V(\Gamma)$ generates in the graph $\Gamma^d$ a connected subgraph* (see [5, Theorem 4.2]). *If, further, the possibility* (1) *from the Main Conjecture holds for that subgraph and* *the group of automorphisms of this subgraph induced by* $N$, *then the possibility* (1) *from the Main Conjecture also holds for* $\Gamma$ *and* $G$ (see [5, Theorem 4.3]). *As a consequence of these results, the Main Conjecture is valid for* $\Gamma$ *and* $G$ *in the case when* $G$ *is a solvable group* (see [5, Theorem 5.1]).

**Remark 2.4.** We correct a few misprints in [5]: on p. $126^{23}$, it should be $\Gamma_1(y) \cap F(G_y) \setminus \{y\}$ instead of $F(G_y) \setminus \{y\}$; on p. $127_{13}$, it should be $(X^-_{k-1} \cup X^+_{k-1})$ instead of $X^+_{k-1}$; on p. $127_{11}$, it should be $X^-_k$ instead of $X^+_{k-1}$, and it should be $(X^-_{k+1} \cup X^+_{k+1})$ instead of $X^-_{k+1}$; on p. $128_{15}$, it should be $V_i$ instead of the last $V$; on p. $129_2$, it should be $x^{g^i}$ instead of $x^{g_i}$; on p. $130^8$, it should be $E^+_{r_j}(\Gamma)$ instead of $E_{r_j}(\Gamma)$; on p. $130^{28}$, it should be $|(G^{\hat{\tau}})_0 : (G^{\hat{\tau}})_M|$ instead of $|(\sigma^{\hat{\tau}})_0 : (\sigma^{\hat{\tau}})_M|$; on p. $131^2$, it should be $x \in y^{\sigma(H)} \setminus \{y\}$ instead of $x \neq y$; on p. $131^7$, it should be $x_m \in y_n^{\sigma(H)}$, $d_\Gamma(x_m, y_n) \leq 1$ instead of $d_\Gamma(x_n, y_m) \geq 1$; on p. $132^{18}$, it should be $\ldots, z_0$ instead of the second $z_0$; on p. $132_5$ and p. $133^5$, it should be $E_0(\Gamma)$ instead of $E(\Gamma)$; on p. $133_4$, it should be $x_3$ instead of $y_3$; on p. $135^9$, it should be $\cong$ instead of the second $=$; on p. $136^{15}$, it should be $F(\bar{g}\bar{h}\bar{g}^{-1})$ instead of $F\bar{g}\bar{h}\bar{g}^{-1}$; on p. $136^{22}$, it should be $\sigma_i/\tau_1 = \sigma((G(0)^{\tau_1})_{\sigma_i/\tau_1})$ instead of $\hat{\sigma_i}/\hat{\tau_1} = \sigma((G^{\hat{\tau_1}})_{\hat{\sigma_i}/\hat{\tau_1}})$; on p. $136^{25}$, it should be $\sigma'/\tau_1 = \sigma((G(0)^{\tau_1})_{\sigma'/\tau_1})$ instead of $\hat{\sigma'}/\hat{\tau_1} = \sigma((G^{\hat{\tau_1}})\hat{\sigma'}/\hat{\tau_1})$.

Note that the proof of (R2) given in [5] closely follows the local approach to the Main Conjecture. The result (R4) is important for us in Section 4.

Among other cases in which the Main Conjecture is valid, we mention only the following one. Recall that $\Gamma$ is a graph with near polynomial growth if there exist positive integers $c$ and $d$ and a sequence of positive integers $n_1 < n_2 < \cdots$ such that the number of vertices of $\Gamma$ in the ball of radius $n_i$ with center $x$ is $\leq cn_i^d$ for all positive integers $i$. Of course, in this case $\Gamma$ is non-hyperbolic. If $\Gamma$ is a graph with near polynomial growth, then the structure of $\Gamma$ is known by [8] and [9]. As a consequence of that description, we have the following.

(R5) *If* $\Gamma$ *is a graph with near polynomial growth, then, for* $\Gamma$ *and* $G$, *the possibility* (1) *from the Main Conjecture holds.*

By [3, Theorem 2.6], if $G_x$ is finite and $G_x^{\Gamma(x)}$ is quasiprimitive, then $G_{x,y}^{[1]}$ is a $p$-group for some prime number $p$. At the same time, it is easy to give an example of $\Gamma$ and $G$ such that, for $x \in V(\Gamma)$, $G_x$ is infinite and $G_x^{\Gamma(x)}$ is a primitive group but $G_x^{[n]}/G_x^{[n+1]}$ is not a group of prime power order for each positive integer $n$. (Take, for example, the regular tree of a finite valency $>3$ as $\Gamma$ and the group of all its automorphisms as $G$.) Nevertheless, in the case of infinite $G_x$, there are some specific phenomena in the distribution of composition factors of $G_x$. We study ones in the following Proposition 2.5 and Corollary 2.7 which are important in the local approach to the Main Conjecture (see, for example, Sections 3 and 5).

Obviously, for any positive integers $m \leq n$, each composition factor of the group $G_x^{[n]}/G_x^{[n+1]}$ is isomorphic to a composition factor of the group $G_x^{[m]}/G_x^{[m+1]}$. It follows from the next Proposition 2.5 (see Corollary 2.7) that the converse also holds in the case when $G_x$ is infinite, $G_x^{\Gamma(x)}$ is a quasiprimitive group and $m > 1$. Note that, in what follows, it is natural to use a terminology and some facts concerning profinite groups (taking into account that the closures of vertex stabilizers are profinite groups), but we avoid this in the present paper.

**Proposition 2.5.** *Suppose $G_x$ is infinite and $G_x^{\Gamma(x)}$ is quasiprimitive. Let $A$ be an abstract simple non-trivial group isomorphic to a composition factor of the group $G_{x,y}^{[1]}/G_x^{[2]}$. Then for any normal non-trivial subgroup $N$ of $G_x$ there exists a positive integer $n$ such that $A$ is isomorphic to a composition factor of the group $(N \cap G_x^{[n]})/(N \cap G_x^{[n+1]})$. In particular, for each positive integer $n$, the group $A$ is isomorphic to a composition factor of the group $G_x^{[n]}/G_x^{[n+1]}$.*

**Proof.** Let $Y$ be the set of $x' \in \Gamma(x)$ such that $(G_{x,y}^{[1]})^{\Gamma(x')}$ has no composition factor isomorphic to $A$. Observe that, by the hypothesis of the proposition, $Y$ is a proper subset of $\Gamma(x)$. It is easy to see that $Y$ is a block of an imprimitivity system $\rho$ of $G_x^{\Gamma(x)}$. (The blocks of $\rho$ can be one-element.) Since $G_x^{\Gamma(x)}$ is a non-trivial quasiprimitive group, the kernel of the action of $G_x^{\Gamma(x)}$ on $\rho$ is trivial. Thus, for any $h \in G_x \setminus G_x^{[1]}$, there exist $y' \in \Gamma(x)$ such that $(G_{x,y'}^{[1]})^{\Gamma(h(y'))}$ has a composition factor isomorphic to $A$.

Without loss we assume that $G$ is a closed subgroup of $Aut(\Gamma)$. A subgroup $H$ of $G_z$, where $z \in V(\Gamma)$, is said to be an $\{A\}'$-group if, for each positive integer $i$, the group $(H \cap G_z^{[i]})/(H \cap G_z^{[i+1]})$ has no composition factor isomorphic to $A$. It is easy to see that this definition is independent of $z$ with the property $H \in G_z$. Any closed subgroup $H$ of $G$ stabilizing some vertex contains a largest normal $\{A\}'$-subgroup which is denoted by $O_{\{A\}'}(H)$. (If $H$ stabilizes a vertex $z$, then $O_{\{A\}'}(H)$ is the intersection of the preimages of the subgroups $O_{\{A\}'}(H/(H \cap G_z^{[i]}))$ of $H/(H \cap G_z^{[i]})$ under the natural homomorphisms $H \to H/(H \cap G_z^{[i]})$ where $i$ runs over the set of positive integers and, for each $i$, $O_{\{A\}'}(H/(H \cap G_z^{[i]}))$ is the largest normal subgroup of $H/(H \cap G_z^{[i]})$ without composition factors isomorphic to $A$.) We must show that $O_{\{A\}'}(G_x) = 1$.

Suppose first that $O_{\{A\}'}(G_x) \leq G_x^{[2]}$. Then $O_{\{A\}'}(G_x)$ is a subnormal $\{A\}'$-subgroup of $G_y$. Hence $O_{\{A\}'}(G_x) \leq O_{\{A\}'}(G_y)$. Since there exists an element in $G$ mapping $x$ to $y$, we also have $O_{\{A\}'}(G_y) \leq O_{\{A\}'}(G_x)$. Thus $O_{\{A\}'}(G_x) = O_{\{A\}'}(G_y)$ is a normal subgroup as of $G_x$ and of $G_y$, and the result follows.

Suppose now that $O_{\{A\}'}(G_x) \not\leq G_x^{[2]}$. If $O_{\{A\}'}(G_x) \not\leq G_x^{[1]}$, let $h$ be an element from $O_{\{A\}'}(G_x) \setminus G_x^{[1]}$. On the other hand, if $O_{\{A\}'}(G_x) \leq G_x^{[1]}$, then $O_{\{A\}'}(G_y)^{\Gamma(x)} \neq 1$, and we let $h \in O_{\{A\}'}(G_y) \setminus G_x^{[1]}$. Note that, in the latter case, $h \in O_{\{A\}'}(G_{x,y})$. As was observed at the beginning of the proof, in both cases there exists $y' \in \Gamma(x)$ such that $(G_{x,y'}^{[1]})^{\Gamma(h(y'))}$ has a composition factor isomorphic to $A$.

Let $K$ be the preimage under the natural homomorphism $G_x^{[1]} \to G_x^{[1]}/G_{x,y'}^{[1]}$ of the largest normal subgroup of $G_x^{[1]}/G_{x,y'}^{[1]}$ which has no composition factor isomorphic to $A$. Then $hKh^{-1}$ is the preimage under the natural homomorphism $G_x^{[1]} \to G_x^{[1]}/G_{x,h(y')}^{[1]}$ of the largest normal subgroup of $G_x^{[1]}/G_{x,h(y')}^{[1]}$ which has no composition factor isomorphic to $A$. Of course, $O_{\{A\}'}(G_x) \cap G_x^{[1]} \leq K$ and $O_{\{A\}'}(G_{x,y}) \cap G_x^{[1]} \leq K$. At the same time, since $G^{[1]}_{x,y'} \trianglelefteq G^{[1]}_x$ and $(G^{[1]}_{x,y'})^{\Gamma(h(y'))}$ has a composition factor isomorphic to $A$, we have $G^{[1]}_{x,y'} \not\leq hKh^{-1}$, that is $h^{-1}G^{[1]}_{x,y'}h \not\leq K$.

Further, since $h$ is an element in $O_{\{A\}'}(G_x)$ or in $O_{\{A\}'}(G_{x,y})$, $[G^{[1]}_{x,y'}, h]$ is a subgroup of $O_{\{A\}'}(G_x) \cap G^{[1]}_x$ or of $O_{\{A\}'}(G_{x,y}) \cap G^{[1]}_x$. In both cases, $[G^{[1]}_{x,y'}, h] \leq K$. Since $G^{[1]}_{x,y'} \leq K$, it follows that $h^{-1}G^{[1]}_{x,y'}h \leq K$, a contradiction.

Thus $O_{\{A\}'}(G_x) = 1$. It remains to show that, for each positive integer $n$, the group $A$ is isomorphic to a composition factor of the group $G^{[n]}_x / G^{[n+1]}_x$. Since $G^{[n]}_x \trianglelefteq G_x$ and $O_{\{A\}'}(G_x) = 1$ by the above, there exists $n' \geq n$ such that $A$ is isomorphic to a composition factor of the group $G^{[n']}_x / G^{[n'+1]}_x$. Since each composition factor of the group $G^{[n']}_x / G^{[n'+1]}_x$ is isomorphic to a composition factor of the group $G^{[n]}_x / G^{[n+1]}_x$, the result follows. □

**Remark 2.6.** It seems that, under hypothesis of Proposition 2.5, the vertices $z$ of $\Gamma$ for which the group $(G^{[d_\Gamma(x,z)]}_x)^{\Gamma(z)}$ has a composition factor isomorphic to $A$ are disposed in some fairly regular way. This is important in the context of the Main Conjecture.

**Corollary 2.7.** *Suppose $G_x$ is infinite and $G^{\Gamma(x)}_x$ is quasiprimitive. Then the set of abstract simple groups isomorphic to composition factors of the group $G^{[n]}_x / G^{[n+1]}_x$ is the same for all $n > 1$.*

## 3. A trial of the local approach to the Main Conjecture

Recall that the local approach to the Main Conjecture consists in fixing a finite permutation group $R$ and considering the Main Conjecture under the additional assumption that $G^{\Gamma(x)}_x$ coincides with $R$ (see the Introduction). In the present section, we prove the Main Conjecture for certain groups $R$ (within the local approach) using simple and rather direct arguments. In the next section we develop techniques which enable us to consider (within the local approach to the Main Conjecture) more interesting and complicated cases of groups $R$.

**Proposition 3.1.** *Suppose that the hypothesis of the Main Conjecture holds. Let $x \in V(\Gamma)$ and $y \in \Gamma(x)$. Suppose that the group $G^{\Gamma(x)}_x$ is transitive. Suppose also that any non-trivial subnormal subgroup $S$ of the group $G^{\Gamma(x)}_{x,y}$ for which $|G^{\Gamma(x)}_{x,y} : N_{G^{\Gamma(x)}_{x,y}}(S)|$ divides some power of $|\Gamma(x)|-1$ acts transitively on $\Gamma(x) \setminus \{y\}$. Then $G_x$ is finite or $\Gamma$ is a regular tree. (In particular, for such $\Gamma$ and $G$ the Main Conjecture is valid.)*

**Proof.** Suppose that $\Gamma$ is not a regular tree. Then there exists a positive integer $s$ such that $G$ is $s$-arc-transitive but not $(s+1)$-arc-transitive. Assume $G^{[s]}_x \neq 1$. Then there is an $s$-arc $(x_0, x_1, \ldots, x_s = x)$ of the graph $\Gamma$ such that the group $G^{[s]}_x$ acts non-trivially on $\Gamma(x_0) \setminus \{x_1\}$. Since $G^{[1]}_{x_1}$ is normal in $G_{x_0,x_1}$, and, for each $2 \leq i \leq s$, $G^{[i]}_{x_i}$ is normal in $G^{[i-1]}_{x_{i-1}}$, we have that $(G^{[s]}_{x_s})^{\Gamma(x_0)}$ is a non-trivial subnormal subgroup of the group $G^{\Gamma(x_0)}_{x_0,x_1}$. Since the group $G_{x_0,x_1,\ldots,x_s}$ normalizes the group $G^{[s]}_{x_s}$, and, for each $2 \leq i \leq s$, $|G_{x_0,x_1,\ldots,x_{i-1}} : G_{x_0,x_1,\ldots,x_i}| = |\Gamma(x)| - 1$ by $s$-arc-transitivity of $G$, we also have that the index in $G_{x_0,x_1}^{\Gamma(x_0)}$ of the normalizer of $(G_{x_s}^{[s]})^{\Gamma(x_0)}$ in $G_{x_0,x_1}^{\Gamma(x_0)}$ divides $(|\Gamma(x)|-1)^s$. By the hypothesis of the proposition, the group $G_{x_s}^{[s]}$ acts transitively on $\Gamma(x_0) \setminus \{x_1\}$. Since $G_{x_s}^{[s]} \leq G_{x_0,x_1,\ldots,x_s}$, it follows that $G$ is $(s+1)$-arc-transitive, contradicting the choice of $s$. Thus $G_x^{[s]} = 1$, and $G_x$ is finite. The result follows. $\square$

**Example 3.2.** Let $\Gamma$ be a connected locally finite graph, and $G$ a vertex-transitive group of automorphisms of $\Gamma$. Suppose the group $G_x^{\Gamma(x)}$, $x \in V(\Gamma)$, contains a normal subgroup which is $PSL_2(q)$ acting in the natural way on $q+1$ points of $PG_1(q)$. We apply Proposition 3.1 to prove that $G_x$ is finite or $\Gamma$ is the regular tree of valency $q+1$. (Thus, for $\Gamma$ and $G$ the Main Conjecture is valid.)

Let $S$ be a non-trivial subnormal subgroup of the group $G_{x,y}^{\Gamma(x)}$, $y \in \Gamma(x)$. Then $S \cap O_p(G_{x,y}^{\Gamma(x)}) \neq 1$ where $p = char(\mathbf{F}_q)$. Therefore in the case when $|G_{x,y}^{\Gamma(x)} : N_{G_{x,y}^{\Gamma(x)}}(S)|$ is a power of $p$ (in which case $N_{G_{x,y}^{\Gamma(x)}}(S)$ acts irreducibly on $O_p(G_{x,y}^{\Gamma(x)})$), $O_p(G_{x,y}^{\Gamma(x)}) \leq S$ and, as a result, $S$ acts transitively on $\Gamma(x) \setminus \{y\}$. Thus, by Proposition 3.1, $G_x$ is finite or $\Gamma$ is the regular tree of valency $q+1$. By the way, mention that in the former case $G_{x,y}^{[1]}$ is a $p$-group, $G_x^{[2]} = 1$ if $p > 3$, $G_x^{[4]} = 1$ if $p = 3$, and $G_x^{[3]} = 1$ if $p = 2$ (see [14]).

**Remark 3.3.** Under the hypothesis of Proposition 3.1, the group $G_x^{\Gamma(x)}$ is regular (in which case, obviously, $G_x^{[1]} = 1$) or doubly transitive. In addition, if $G_x^{\Gamma(x)}$ is doubly transitive, then $G_x^{\Gamma(x)}$ cannot contain a normal subgroup which is $PSL_n(q)$, $n > 2$, acting in the natural way on the set of points of $PG_{n-1}(q)$ (since otherwise the non-trivial normal subgroup $O_p(G_{x,y}^{\Gamma(x)})$ of $G_{x,y}^{\Gamma(x)}$, where $p = char(\mathbf{F}_q)$, acts intransitively on $\Gamma(x) \setminus \{y\}$). By known results on the FVSRP (see [16]), it follows that, under the hypothesis of Proposition 3.1, $G_x^{[4]} = 1$ in the case when $G_x$ is finite.

**Remark 3.4.** It follows from Proposition 3.1 and the classification of finite doubly transitive permutation groups (see, for example, [1]) that, for $\Gamma$ and $G$ satisfying the hypothesis of the Main Conjecture and for $x \in V(\Gamma)$, if $G_x^{\Gamma(x)}$ is doubly transitive (equivalently, $G$ is 2-arc-transitive), $G_x$ is infinite and $\Gamma$ is not a tree, then only cases 1 (with $q > 2$), 2 (with $q > 2$), 4, 5, 6 from Table 7.3 of [1] and cases 2 (with $d > 2$), 5, 6, 7, 16 from Table 7.4 of [1] are possible for $G_x^{\Gamma(x)}$.

**Proposition 3.5.** *Suppose that the hypothesis of the Main Conjecture holds. Let $x \in V(\Gamma)$ and $y \in \Gamma(x)$. Suppose that the group $G_x^{\Gamma(x)}$ is transitive. Let $A$ be an abstract simple non-trivial group. Suppose that $A$ is isomorphic to a composition factor of the group $G_{x,y}^{[1]}/G_x^{[2]}$. Suppose also that any subnormal subgroup of the group $G_{x,y}^{\Gamma(x)}$ with a composition factor isomorphic to $A$ acts transitively on $\Gamma(x) \setminus \{y\}$. Then $G_x$ is finite or $\Gamma$ is a regular tree. (In particular, for such $\Gamma$ and $G$ the Main Conjecture is valid.)*

**Proof.** Assume $G_x$ is infinite. It follows from the hypothesis of the proposition that $G_x^{\Gamma(x)}$ is a doubly transitive group. Since, in addition, $A$ is isomorphic to a composition factor of the group $G_{x,y}^{[1]}/G_x^{[2]}$, Proposition 2.5 gives that, for each positive integer $n$, the group $A$ is isomorphic to a composition factor of $G_x^{[n]}/G_x^{[n+1]}$.

Suppose that $\Gamma$ is not a regular tree. Then there exists a positive integer $s$ such that $G$ is $s$-arc-transitive but not $(s+1)$-arc-transitive. By the above, there exists an $s$-arc $(x_0, x_1, \ldots, x_s = x)$ of the graph $\Gamma$ such that the group $(G_x^{[s]})^{\Gamma(x_0)}$ has a composition factor isomorphic to $A$. Since $(G_{x_s}^{[s]})^{\Gamma(x_0)}$ is a subnormal subgroup of the group $G_{x_0,x_1}^{\Gamma(x_0)}$, it follows that the subgroup $G_{x_s}^{[s]}$ of the group $G_{x_0,x_1,\ldots,x_s}$ acts transitively on $\Gamma(x_0) \setminus \{x_1\}$, contradicting the choice of $s$. $\square$

**Remark 3.6.** Under the hypothesis of Proposition 3.5, the group $G_x^{\Gamma(x)}$ is doubly transitive. By known results on the FVSRP (see [10]), it follows that, under the hypothesis of Proposition 3.5, $G_x^{[6]} = 1$ in the case where $G_x$ is finite.

Whereas Proposition 3.1 eliminates some easy cases of groups $R$ in the local approach to the Main Conjecture, the Proposition 3.5 rather restricts the possible structure of $G_x$ in some more complicated cases of $R$ in the approach. We demonstrate such use of Proposition 3.5 with the following example.

**Example 3.7.** Let $\Gamma$ be a connected locally finite graph, $G$ a vertex-transitive group of automorphisms of $\Gamma$, $x \in V(\Gamma)$ and $y \in \Gamma(x)$. Suppose that $G_x^{\Gamma(x)}$ contains a normal subgroup $L$ such that one of the following holds:

(a) $L$ is isomorphic to $PSL_n(q)$, $n > 2$, $q$ a power of a prime $p$, acting in the natural way on $(q^n - 1)/(q - 1)$ points of $PG_{n-1}(q)$;
(b) $L$ contains a regular normal elementary abelian subgroup $V$ of order $q^n$, $n > 1$, $q$ a power of a prime $p$ and $(n, q) \neq (2, 2)$, and $L_y$ is isomorphic to $SL_n(q)$ and acts on $V$ in the natural way;
(c) $L$ contains a regular normal elementary abelian subgroup $V$ of order $q^{2n}$, $n > 1$, $q$ a power of a prime $p$, and $L_y$ is isomorphic to $Sp_{2n}(q)$ and acts on $V$ in the natural way.

We apply Proposition 3.5 to restrict the possible structure of $G_x$ under these assumptions. In the second part of the present paper we will continue to consider the Main Conjecture in the cases (a), (b) and (c).

In all cases (a), (b) and (c), $G_x^{\Gamma(x)}$ is doubly transitive. In case (a), for any $x' \in V(\Gamma)$, $\Gamma(x')$ can be identified in the natural way with the set of points of the projective space $PG_{n-1}(q)$. In both cases (b) and (c), the action of $L_y$ on $V$ determines on $V$ a structure of $n$-dimensional vector space over $\mathbf{F}_q$ denoted by $V_x^y$, and $\Gamma(x)$ can be identified in the natural way with the corresponding affine space $AG_n(q)$ (which is independent of the choice of $y$). Obviously, this is valid for any $x' \in V(\Gamma)$ and $y' \in \Gamma(x')$ instead of $x$ and $y$ as well. Next for arbitrary $x' \in V(\Gamma)$ and $y' \in \Gamma(x')$, we write $[x' : y']$ for the set of lines containing $y'$ of the projective space $\Gamma(x')$ in case (a) or of the affine space $\Gamma(x')$ in both cases (b) and (c).

If (a) holds, then the group $O_p(G_{x,y}^{\Gamma(x)}) = O_p(L_y)$ is elementary abelian of order $q^{n-1}$, and its orbits on $\Gamma(x) \setminus \{y\}$ are the sets $X \setminus \{y\}$, where $X$ runs over the set $[x : y]$ of lines of the projective space $\Gamma(x)$ containing $y$. The kernel $K$ of the action of $G_{x,y}^{\Gamma(x)}$ on $[x : y]$ is a semidirect product of $O_p(L_y)$ by a cyclic group $C$ of order dividing

$q-1$. If $(n, q) \neq (3, 2), (3, 3)$, each subnormal subgroup of the group $G_{x,y}^{\Gamma(x)}$ either has a composition factor isomorphic to $PSL_{n-1}(q)$ and acts transitively on $\Gamma(x) \setminus \{y\}$ or is contained in $K$. (If $(n, q) = (3, 2)$, each subnormal subgroup of the group $G_{x,y}^{\Gamma(x)}$ either has a composition factor isomorphic to the cyclic group of order 3 and acts transitively on $\Gamma(x) \setminus \{y\}$ or is contained in $K$; if $(n, q) = (3, 3)$, each subnormal subgroup of the group $G_{x,y}^{\Gamma(x)}$ inducing on the set $[x : y]$ a group with a composition factor isomorphic to the cyclic group of order 3 acts transitively on $\Gamma(x) \setminus \{y\}$.)

If (b) holds and $(n, q) \neq (2, 3)$, it is easy to see that each subnormal subgroup of the group $G_{x,y}^{\Gamma(x)}$ either has a composition factor isomorphic to $PSL_n(q)$ and acts transitively on $\Gamma(x) \setminus \{y\}$ or is contained in $Z(GL(V_x^y))$. (If (b) holds and $(n, q) = (2, 3)$, each subnormal subgroup of $G_{x,y}^{\Gamma(x)}$ with a composition factor isomorphic to the cyclic group of order 3 acts transitively on $\Gamma(x) \setminus \{y\}$.) Analogously, if (c) holds and $(n, q) \neq (2, 2)$, each subnormal subgroup of the group $G_{x,y}^{\Gamma(x)}$ either has a composition factor isomorphic to $PSp_{2n}(q)$ and acts transitively on $\Gamma(x) \setminus \{y\}$ or is contained in $Z(GL(V_x^y))$. (If (c) holds and $(n, q) = (2, 2)$, each non-trivial subnormal subgroup of $G_{x,y}^{\Gamma(x)}$ acts transitively on $\Gamma(x) \setminus \{y\}$.)

Further, if $G_x$ is finite, then we have $G_x^{[6]} = 1$ in case (a) (see [10]), and $G_{x,y}^{[1]} = 1$ in both cases (b) and (c) (see [15]). Suppose now that $G_x$ is infinite and $\Gamma$ is not a regular tree.

If (a) holds and $(n, q) \neq (3, 3)$, then Proposition 3.5 (with $A$ isomorphic to $PSL_{n-1}(q)$ in the case $(n, q) \neq (3, 2)$ and with $A$ isomorphic to the cyclic group of order 3 in the case $(n, q) = (3, 2)$) implies that, for any $z \in \Gamma(y) \setminus \{x\}$, $(G_{y,z}^{[1]})^{\Gamma(x)} \leq K$. By arguments going back to [13] and [17], it follows that (in the case $(n, q) \neq (3, 3)$) either $G$ is 3-arc-transitive and the stabilizer in $G_{x,y,z}$ of a line from $[x : y]$ coincides with the stabilizer in $G_{x,y,z}$ of a line from $[z : y]$, or $G$ is 2-arc-transitive and the stabilizer in $G_{x,y}$ of a line from $[x : y]$ coincides with the stabilizer in $G_{x,y}$ of a line from $[y : x]$ or of a hyperplane of $\Gamma(y)$ containing $x$. (Thus we have analogs of so-called $n = 3$, 3-arc-transitive, collineation and correlation cases in the FVSRP with the same $R$; see [10].)

If (b) holds and $(n, q) \neq (2, 3)$, then Proposition 3.5 (with $A$ isomorphic to $PSL_n(q)$) implies that, for any $z \in \Gamma(y) \setminus \{x\}$, $(G_{y,z}^{[1]})^{\Gamma(x)} \leq Z(GL(V_x^y))$. In particular, $G$ is not 4-arc-transitive.

Finally, if (c) holds, then $q > 2$ by Proposition 3.1. Now Proposition 3.5 (with $A$ isomorphic to $PSp_{2n}(q)$) implies that, for any $z \in \Gamma(y) \setminus \{x\}$, $(G_{y,z}^{[1]})^{\Gamma(x)} \leq Z(GL(V_x^y))$. In particular, $G$ is not 4-arc-transitive.

However, to continue to consider the Main Conjecture in the cases (a), (b) and (c) we need some results and techniques from Sections 4 and 5. We will continue to consider the Main Conjecture in the cases (a), (b) and (c) in the second part of the present paper.

Propositions 3.1 and 3.5 eliminate some easy cases (and situations) in the local approach to the Main Conjecture. To eliminate more complicated cases we need more delicate techniques. In the subsequent sections of the present paper, we develop a method allowing us to consider many of those more complicated cases. In view of an analogy of this method with the track method for the FVSRP we indicate it as the modified track method.

## 4. Background for the modified track method

The modified track method is based on some general results which are proved in this section.

**Theorem 4.1.** *Let $\Gamma$ be a connected locally finite graph, and $G$ a vertex-transitive group of automorphisms of $\Gamma$. Let $g \in G$, and $(\ldots, x_1, x_0, x_1, \ldots)$ be a generalized $g$-track of $\Gamma$. Let $H = \langle G_{x_0,x_1,\ldots}, g\rangle$, let $K$ be the normal closure in $H$ of $G_{x_0,x_1,\ldots}$, and let $X$ be the $H$-orbit containing $x_i$ for all integers $i$. Then the following assertions hold.*

(1) *The subgraph of the graph $\Gamma^{d_\Gamma(x_0,x_1)}$ generated by $X$ is connected.*
(2) *Each $K$-orbit on $X$ is of the form $K(x_i)$ for some integer $i$. If $i$, $j$ are integers and $x_i \neq x_j$, then $K(x_i) \neq K(x_j)$. The group induced by $H$ on the set of $K$-orbits on $X$ is a transitive group of order $|\{x_i : i \in \mathbf{Z}\}|$ and is generated by the element induced by $g$.*
(3) *If $K$-orbits on $V(\Gamma)$ are infinite, then the subgraph of the graph $\Gamma^{d_\Gamma(x_0,x_1)}$ generated by $X$ is hyperbolic. In particular, if $\Gamma$ is non-hyperbolic, then $K$-orbits on $V(\Gamma)$ are finite.*
(4) *If $\Gamma$ is non-hyperbolic and $G$ is a closed subgroup of $Aut(\Gamma)$, then there exists a positive integer $l$ such that, for any integer $i$, both the group $G_{x_i,x_{i+1},\ldots,x_{i+l}}$ and the stabilizer in $G$ of the $K$-orbits $K(x_i), K(x_{i+1}), \ldots, K(x_{i+l})$ stabilize each $K$-orbit on $X$.*

**Proof.** (1) Let $\Delta$ be the subgraph of the graph $\Gamma^{d_\Gamma(x_0,x_1)}$ generated by $X$, and let $X'$ be the connected component of the graph $\Delta$ containing $x_0$. Obviously, the restriction of $H$ to $X$ is a subgroup of $Aut(\Delta)$. Since $g(x_0) = x_1 \in X'$, it follows that the group $\langle G_{x_0,x_1,\ldots}, g\rangle = H$ stabilizes the set $X'$. Since $X$ is an $H$-orbit, this gives $X' = X$ completing the proof of (1).

(2) If the set $\{x_i : i \in \mathbf{Z}\}$ is finite, then $K$ stabilizes each vertex from $X = \{x_i : i \in \mathbf{Z}\}$ and (2) holds. Hence we may assume that $x_i \neq x_j$ for any integers $i \neq j$.

Since $H = \langle K, g\rangle$, where $K \trianglelefteq H$, and $H$ acts transitively on $X$, the group $H$ induces on the set of $K$-orbits on $X$ a transitive cyclic group which is generated by the element induced by $g$. As a result, each $K$-orbit on $X$ is of the form $g^i(K(x_0)) = K(g^i(x_0)) = K(x_i)$ for some integer $i$. Suppose that $K(x_{i'}) = K(x_{j'})$ for some integers $i' \geq j'$. Let $h$ be an element of $K$ such that $h(x_{i'}) = x_{j'}$. Then $g^{i'-j'}h(x_{i'}) = x_{i'}$. At the same time, since the element $h$ (being an element in $K$) fixes vertices $x_i$ for all sufficiently large integers $i$, say for all $i \geq i_0$ where $i_0 \in \mathbf{Z}$, we have $g^{i'-j'}h(x_i) = x_{i+i'-j'}$ for all integers $i \geq i_0$. Hence $d_\Gamma(x_{i'}, x_{i_0}) = d_\Gamma((g^{i'-j'}h)^k(x_{i'}), (g^{i'-j'}h)^k(x_{i_0})) = d_\Gamma(x_{i'}, x_{i_0+k(i'-j')})$ for all positive integers $k$. Since the graph $\Gamma$ is connected and locally finite, it follows that $i' = j'$. The proof of (2) is complete.

(3) Since $K$-orbits on $V(\Gamma)$ are infinite, the set $\{x_i : i \in \mathbf{Z}\}$ is infinite. By (1) and (2), it follows the subgraph $\Delta$ of the graph $\Gamma^{d_\Gamma(x_0,x_1)}$ generated by $X$ is connected, $H^X$ is a vertex-transitive group of automorphisms of $\Delta$, $K^X \trianglelefteq H^X$, and each $K^X$-orbit is of the form $K(x_i)$, $i \in \mathbf{Z}$, with $K(x_i) \cap K(x_j) = \emptyset$ for $j \neq i$. In addition, $H^X$ induces on the set of $K^X$-orbits an infinite cyclic group coinciding with the group induced by $\langle g\rangle$.

Suppose that $\Delta$ is non-hyperbolic. Then, by [5, Theorem 4.2] (see (R4) in Section 2), there exists a positive integer $d$ such that the $K$-orbit $K(x_0)$ generates in the graph $\Delta^d$ a connected subgraph. Let $\Delta^d(x_0) \cap K(x_0) = \{y_1, \ldots, y_m\}$. For each $1 \leq k \leq m$, there exists an element $h_k$ in $K$ such that $h_k(x_0) = y_k$. Since $K(x_0)$ generates a connected subgraph in $\Delta^d$, it follows that $K(x_0) = \langle h_1, \ldots, h_m \rangle(x_0)$. Since each element in $K$ stabilizes vertices $x_i$ for all sufficiently large $i$, there exists a vertex $x_{i'}$, $i' \in \mathbf{Z}$, which is stabilized by $\langle h_1, \ldots, h_m \rangle$. Since the graph $\Gamma$ is connected and locally finite, it follows that the $K$-orbit $K(x_0) = \langle h_1, \ldots, h_m \rangle(x_0)$ is finite, a contradiction. The proof of (3) is complete.

(4) Without loss we may assume that the set $\{x_i : i \in \mathbf{Z}\}$ is infinite. By (2), each $K^X$-orbit is of the form $K(x_i)$, $i \in \mathbf{Z}$, with $K(x_i) \cap K(x_j) = \emptyset$ for $j \neq i$, and $H^X$ induces on the set of $K^X$-orbits an infinite cyclic group coinciding with the group induced by $\langle g \rangle$. By (3), $K$-orbits on $V(\Gamma)$ are finite.

For any integers $i \leq j$, let $S[i, j]$ be the stabilizer in $G$ of the $K$-orbits $K(x_i), \ldots, K(x_j)$. For any integer $i$, put

$$S[i] := \cap_{j \geq i} S[i, j].$$

In addition, put

$$S := \cap_{i \in \mathbf{Z}} S[i].$$

Since the graph $\Gamma$ is non-hyperbolic, $G$ is a closed subgroup of $Aut(\Gamma)$ and $K$-orbits on $V(\Gamma)$ are finite, it follows from [7, Theorem 2] (see (R1) in Section 2) that, for any integers $i \leq j$,

$$|S[i, j] : S[i-1, j]| = |S[i, j] : gS[i-1, j]g^{-1}| = |S[i, j] : S[i, j+1]|.$$

As a result, if $S[i, j] = S[i-1, j]$ or $S[i, j] = S[i, j+1]$ for some integers $i \leq j$, then $S[i, j] = S$.

Suppose that $S[0, j] \neq S[-1, j]$ for all non-negative integers $j$. Since $K$-orbits are finite and $G$ is a closed subgroup of $Aut(\Gamma)$, it follows that $S[0] \neq S[-1]$. Thus there exists an element $h$ in $S[0]$ which does not stabilize the $K$-orbit $K(x_{-1})$. Next since $G$ is a closed subgroup of $Aut(\Gamma)$, there exists a positive integer $r$ such that, for any integer $i$, the $G_{x_{i-r}, \ldots, x_{i-1}, x_i}$-orbit containing $x_{i+1}$ coincides with the $G_{\ldots, x_{i-1}, x_i}$-orbit containing $x_{i+1}$. For each non-negative integer $n$, the element $g^{-n}hg^n$ stabilizes the finite set $\cup_{0 \leq s \leq r} K(x_s)$. Hence there exist integers $n'' > n' \geq 0$ such that the restrictions of $g^{-n''}hg^{n''}$ and $g^{-n'}hg^{n'}$ to the set $\cup_{0 \leq s \leq r} K(x_s)$ coincide. Thus the element $h' := g^{n'-n''}h^{-1}g^{n''-n'}h$ stabilizes pointwise the set $\cup_{n' \leq s \leq n'+r} K(x_s)$. Observe that $h' \in S[0]$ (since both $g^{n'-n''}h^{-1}g^{n''-n'}$ and $h$ are in $S[0]$) and $h'(K(x_{-1})) \neq K(x_{-1})$ (since $h(K(x_{-1})) \neq K(x_{-1})$ while $g^{n'-n''}h^{-1}g^{n''-n'}$ stabilizes the set $K(x_{-1})$). Furthermore, replacing (in case of need) the element $h'$ by the element $a^{-1}h'a$ for an appropriate $a \in K$, we may assume that $h'(x_{-1}) \notin K(x_{-1})$.

Since $h'(x_i) = x_i$ for all $n' \leq i \leq n' + r$, the choice of $r$ yields that there exists an element $h_1 \in G_{\ldots, x_{n'+r-1}, x_{n'+r}}$ such that $h'h_1(x_{n'+r+1}) = x_{n'+r+1}$. Now, for $h'_1 := h'h_1$, we have $h'_1(x_i) = x_i$ for all $n' \leq i \leq n'+r+1$ and $h'_1(x_{-1}) = h'(x_{-1}) \notin K(x_{-1})$. Inductively, suppose that, for a positive integer $t$, we have an element $h'_t$ in $G$ such that $h'_t(x_i) = x_i$ for all $n' \leq i \leq n' + r + t$ and $h'_t(x_{-1}) \notin K(x_{-1})$. The choice of $r$ yields that there exists an element $h_{t+1} \in G_{\ldots, x_{n'+r+t-1}, x_{n'+r+t}}$ such that $h'_t h_{t+1}(x_{n'+r+t+1}) = x_{n'+r+t+1}$. Put $h'_{t+1} := h'_t h_{t+1}$. Then we have $h'_{t+1}(x_i) = x_i$ for all $n' \leq i \leq n' + r + t + 1$ and $h'_{t+1}(x_{-1}) = h'_t(x_{-1}) \notin K(x_{-1})$.

Thus for any positive integer $t$, there exists an element $h'_t$ in $G$ such that $h'_t(x_i) = x_i$ for all $n' \leq i \leq n' + r + t$ and $h'_t(x_{-1}) \notin K(x_{-1})$. Since $G$ is a closed subgroup of $Aut(\Gamma)$, the group $G_{x_{n'}}$ is compact (see Section 2). Thus the sequence $h'_1, h'_2, \ldots$ of elements in $G_{x_{n'}}$ has a limit point $\overline{h} \in G_{x_{n'}}$. Obviously, $\overline{h}(x_i) = x_i$ for all $i \geq n'$ and $\overline{h}(x_{-1}) \notin K(x_{-1})$. But $G_{x_{n'}, x_{n'+1}, \ldots} = g^{n'} G_{x_0, x_1, \ldots} g^{-n'} \leq K$. Thus $\overline{h} \in K$, contradicting $\overline{h}(x_{-1}) \notin K(x_{-1})$.

Hence there exists a positive integer $l_1$ such that $S[0, l_1] = S[-1, l_1]$. As it was observed before, this gives $S[0, l_1] = S$.

Assume that $G_{x_0, x_1, \ldots, x_j} \nleq S$ for each positive integer $j$. Since $S = S[0, l_1]$, it follows that, for each positive integer $j$, there exists an element $b_j \in G_{x_0, x_1, \ldots, x_j}$ such that $b_j(K(x_0)) \neq K(x_0)$. (In fact, if $G_{x_0, x_1, \ldots, x_j} \leq S[0, 0]$ for some positive integer $j$, then $G_{x_0, x_1, \ldots, x_{j+l_1}} \leq S[0, l_1] = S$, contradicting the assumption.) Since $G_{x_0}$ is a compact group, the sequence $b_1, b_2, \ldots$ of elements in $G_{x_0}$ has a limit point $\overline{b} \in G_{x_0}$. Since the $K$-orbit $K(x_0)$ is finite and $b_j(K(x_0)) \neq K(x_0)$ for each positive integer $j$, we have $\overline{b}(K(x_0)) \neq K(x_0)$. On the other hand, since $b_j \in G_{x_0, x_1, \ldots, x_j}$ for each positive integer $j$, we have $\overline{b} \in G_{x_0, x_1, \ldots} \leq K$. Thus we get a contradiction. Hence there exists a positive integer $l_2$ such that $G_{x_0, x_1, \ldots, x_{l_2}} \leq S$.

Put $l := \max\{l_1, l_2\}$. Then, for any integer $i$, we have

$$G_{x_i, x_{i+1}, \ldots, x_{i+l}} = g^i G_{x_0, x_1, \ldots, x_l} g^{-i} \leq g^i G_{x_0, x_1, \ldots, x_{l_2}} g^{-i} \leq g^i S g^{-i} = S$$

and

$$S[i, i+l] = g^i S[0, l] g^{-i} \leq g^i S[0, l_1] g^{-i} \leq S.$$

The proof of the theorem is complete. $\square$

**Remark 4.2.** In the notation of Theorem 4.1, suppose that the $K$-orbits on $V(\Gamma)$ are finite. (By the assertion (3) of the theorem, this holds, in particular, in the case when $\Gamma$ is non-hyperbolic.) Suppose also that the set $\{x_i : i \in \mathbf{Z}\}$ is infinite. Let $Q = Aut(\Gamma)_{\{X\}}$ be the stabilizer in $Aut(\Gamma)$ of the set $X$, and let $\tilde{Q} = Q^X$ be the group induced by $Q$ on $X$. Then the set of $K$-orbits on $X$ is an imprimitivity system of the group $\tilde{Q}$. Moreover, the group induced by $\tilde{Q}$ on this imprimitivity system is either infinite cyclic generated by the element induced by $g$ or infinite dihedral generated by the element induced by $g$ and by the element mapping $K(x_i)$ to $K(x_{-i})$ for each integer $i$. These assertions follow from Theorem 4.1 and a very special case of [6, Corollary 1]. In fact, let $\tilde{H} = H^X$ and $\tilde{K} = K^X$. By Theorem 4.1, $\tilde{H}$ is a vertex-transitive group of bounded automorphisms of the subgraph $\Delta$ of the graph $\Gamma^{d_\Gamma(x_0, x_1)}$ generated by $X$. In addition, $\tilde{K}$ is the set of those bounded automorphisms of finite order of $\Delta$ which are in $\tilde{H}$. By [6, Corollary 1], it follows that $\tilde{K}$-orbits on $X$ are also the orbits on $X$ of the normal subgroup of $Aut(\Delta)$ consisting of all bounded automorphisms of $\Delta$ of finite order. In addition, by [6, Corollary 1], the group $Aut_0(\Delta)$ of all bounded automorphisms of $\Delta$, which is also a normal subgroup in $Aut(\Delta)$, induces on the set of $K$-orbits on $X$ the transitive infinite cyclic group generated by the element induced by $g$. Since $\tilde{H} \leq \tilde{Q} \leq Aut(\Delta)$, the result follows. In particular, for $\Delta$ and $\tilde{Q}$ (and also for $\Delta$ and $Aut(\Delta)$) the possibility (1) from the Main Conjecture holds.

**Remark 4.3.** In the notation of Theorem 4.1, $K$-orbits on $V(\Gamma)$ are finite if and only if some (equivalently, each) $K$-orbit on $V(\Gamma)$ generates a connected subgraph in the graph

$\Gamma^d$ for some positive integer $d$ (depending, in general, on the $K$-orbit). In fact, the assertion obviously holds in the case where $K$-orbits on $V(\Gamma)$ are finite. On the other hand, if the $K$-orbit $K(x_0)$ generates a connected subgraph in the graph $\Gamma^d$ for some positive integer $d$, then arguments from the proof of the assertion (3) of the theorem can be easily adapted (by replacing $\Delta^d(x_0) \cap K(x_0) = \{y_1, \ldots, y_m\}$ by $\Gamma^d(x_0) \cap K(x_0) = \{y_1, \ldots, y_m\}$) to prove that $K(x_0)$ is finite. Since either each or none of the $K$-orbits on $V(\Gamma)$ generates a connected subgraph in the graph $\Gamma^d$ for some, depending on the $K$-orbit, positive integer $d$, the result follows.

**Remark 4.4.** It follows from the proof of the assertion (4) of the theorem that, in (4) of the theorem, it is sufficient to assume that $\mathrm{Mod}_G(g) = 1$ and $K$-orbits on $V(\Gamma)$ are finite instead of assuming that $\Gamma$ is non-hyperbolic.

To formulate a corollary of Theorem 4.1 we need few definitions. Let $\Gamma$ be a graph, and let $G \le Aut(\Gamma)$.

For a generalized $G$-track $T = (\ldots, x_{-1}, x_0, x_1, \ldots)$ of $\Gamma$, put $H_{T,G} := \langle G_{x_0,x_1,\ldots}, g\rangle$ where $g$ is an element in $G$ such that $T$ is a generalized $g$-track of $\Gamma$. Observe that $H_{T,G}$ is independent of the choice of $g$ in $G$ with this property. (In fact, if $T$ is also a generalized $g'$-track where $g' \in G$, then $g^{-1}g' \in G_{\ldots,x_{-1},x_0,x_1,\ldots}$ and hence $\langle G_{x_0,x_1,\ldots}, g'\rangle = \langle G_{x_0,x_1,\ldots}, g\rangle$.) Let $K_{T,G}$ denote the normal closure of $G_{x_0,x_1,\ldots}$ in $H_{T,G}$.

For a generalized $G$-track $T = (\ldots, x_{-1}, x_0, x_1, \ldots)$ of $\Gamma$, we define $[T]_G$, the $G$-*envelope* of $T$, to be the $H_{T,G}$-orbit containing $x_i$ for all integers $i$.

Further we say that two generalized $G$-tracks $T_1 = (\ldots, x_{1,-1}, x_{1,0}, x_{1,1}, \ldots)$ and $T_2 = (\ldots, x_{2,-1}, x_{2,0}, x_{2,1}, \ldots)$ of $\Gamma$ are $G$-*equivalent* if there exists an integer $k$ such that $K_{T_1,G}(x_{1,j}) = K_{T_2,G}(x_{2,j+k})$ for all integers $j$. Obviously, the $G$-equivalence is an equivalence relation on the set of generalized $G$-tracks of $\Gamma$.

**Corollary 4.5.** *Let $\Gamma$ be a connected locally finite non-hyperbolic graph, and $G$ a vertex-transitive closed subgroup of $Aut(\Gamma)$. Then the following assertions hold.*

(1) *For any generalized $G$-track $T$ of $\Gamma$, $[T]_G = [T^{-1}]_G$.*
(2) *Let $T_1, T_2$ be infinite generalized $G$-tracks of $\Gamma$ such that $[T_1]_G = [T_2]_G$. Then either $T_1$ and $T_2$ or $T_1$ and $T_2^{-1}$ are $G$-equivalent.*
(3) *There exists a positive integer $l_G$ with the following property. If $T_1 = (\ldots, x_{1,-1}, x_{1,0}, x_{1,1}, \ldots)$ and $T_2 = (\ldots, x_{2,-1}, x_{2,0}, x_{2,1}, \ldots)$ are $G$-tracks of $\Gamma$ such that $x_{1,i} = x_{2,i}$ for all $0 \le i \le l_G$, then $T_1$ and $T_2$ are $G$-equivalent.*
(4) *For $x \in V(\Gamma)$, the group $G_x^{[\lfloor \frac{l_G+1}{2} \rfloor]}$, where $\lfloor \frac{l_G+1}{2} \rfloor$ is the largest integer $\le \frac{l_G+1}{2}$, stabilizes all $G$-envelopes of $G$-tracks containing $x$.*
(5) *For $x \in V(\Gamma)$, there are only finitely many pairwise non-$G$-equivalent $G$-tracks of $\Gamma$ passing through $x$.*
(6) *For $x \in V(\Gamma)$, there are only finitely many, say $t_G$, $G$-envelopes of $G$-tracks of $\Gamma$ containing $x$.*
(7) *If $T$ is a $G$-track of $\Gamma$ and $(\ldots, x_{-1}, x_0, x_1, \ldots)$ is a $G$-track of $\Gamma$ such that $x_i \in [T]_G$ for all $0 \le i \le t_G$, then $x_i \in [T]_G$ for all $i \in \mathbf{Z}$.*

**Proof.** (1) Let $T = (\ldots, x_1, x_0, x_1, \ldots)$ be a generalized $g$-track of $\Gamma$, $g \in G$. By Theorem 4.1(4), there exists a positive integer $l$ such that the $\langle G_{x_0,x_1,\ldots}, g\rangle$-orbit

$[T]_G$ is also a $\langle G_{x_0,x_1,\ldots,x_l}, g\rangle$-orbit and the $\langle G_{\ldots,x_{-1},x_0}, g\rangle$-orbit $[T^{-1}]_G$ is also a $\langle G_{x_{-l},\ldots,x_{-1},x_0}, g\rangle$-orbit. Since $\langle G_{x_0,x_1,\ldots,x_l}, g\rangle = \langle G_{x_{-l},\ldots,x_{-1},x_0}, g\rangle$, the result follows.

(2) The assertion (2) follows from arguments in Remark 4.2.

(3) Suppose that the assertion (3) is false. Let $x$ be a vertex of $\Gamma$. Then, for each positive integer $n$, there exist a $g'_n$-track $T'_n = (\ldots, x'_{n,-1}, x'_{n,0}, x'_{n,1}, \ldots)$ of $\Gamma$, $g'_n \in G$, and a $g''_n$-track $T''_n = (\ldots, x''_{n,-1}, x''_{n,0}, x''_{n,1}, \ldots)$ of $\Gamma$, $g''_n \in G$, such that $x = x'_{n,0} = x''_{n,0}$, $x'_{n,i} = x''_{n,i}$ for all $0 < i \le n$, and $T'_n$ and $T''_n$ are not $G$-equivalent. Since $G$ is a closed subgroup of $Aut(\Gamma)$, the sequence $(g''_n)_{n>0}$ contains a subsequence which converges to an element $g$ in $G$. For each integer $i$, put $x_i := g^i(x)$. It is easy to see that $T := (\ldots, x_{-1}, x_0, x_1, \ldots)$ is an infinite $g$-track of $\Gamma$ and, for each positive integer $s$, there exists a positive integer $n_s$ such that $x_i = x'_{n_s,i} = x''_{n_s,i}$ for all $0 \le i \le s$.

By Theorem 4.1(4), there exists a positive integer $l$ such that the group $G_{x_0,x_1,\ldots,x_l}$ stabilizes the set $[T]_G$. In addition, since $G$ is a closed subgroup of $Aut(\Gamma)$, there exists a positive integer $r$ such that the $G_{x_1,\ldots,x_r}$-orbit containing $x_0$ and the $G_{x_1,x_2,\ldots}$-orbit containing $x_0$ coincide. Put $m := \max\{l + 1, r\}$.

Assume we have an element $g'$ in $G$ and a $g'$-track $T' = (\ldots, x'_{-1}, x'_0, x'_1, \ldots)$ of $\Gamma$ such that $x'_i = x_i$ for all $0 \le i \le m$. We claim that under this assumption the $G$-tracks $T'$ and $T$ are $G$-equivalent. By Theorem 4.1(4), there exists a positive integer $l'$ such that the group $G_{x'_0,x'_1,\ldots,x'_{l'}}$ stabilizes the set $[T']_G$. Of course we may assume that $l' \ge m$. First we show that there exists an element $h$ in $G$ such that $h(x'_i) = x_i$ for all $0 \le i \le l' + 1$. It is sufficient to prove that, for each non-negative integer $j$, there exists $h_j \in G$ such that $h_j(x'_i) = x_i$ for all $0 \le i \le m + j$. We construct elements $h_j$ for all non-negative integers $j$ inductively. Obviously, we can take $h_0 = 1$. Suppose now that we have a required element $h_j$ for some non-negative integer $j$. Then $gh_jg'^{-1}(x'_i) = x_i$ for all $1 \le i \le m + j + 1$. Since $m \ge r$ and $gh_jg'^{-1}(x_i) = x_i$ for all $1 \le i \le m$, there exists $h'_j \in G_{x_1,x_2,\ldots}$ such that $h'_j(gh_jg'^{-1}(x_0)) = x_0$. Now $h'_jgh_jg'^{-1}(x_i) = x_i$ for all $0 \le i \le m + j + 1$. Thus we can take $h_{j+1} = h'_jgh_jg'^{-1}$, completing the induction arguments. Hence $h$ exists. Next since the set $[T']_G$ is stabilized by $G_{x'_0,x'_1,\ldots,x'_{l'}}$ and by $g'$, the set $h([T']_G)$ is stabilized by $hG_{x'_0,x'_1,\ldots,x'_{l'}}h^{-1} = G_{x_0,x_1,\ldots,x_{l'}}$ and by $hg'h^{-1}$. Since $g^{-1}hg'h^{-1} \in G_{x_0,x_1,\ldots,x_{l'}}$, it follows that the set $h([T']_G)$ is also stabilized by $g$. But, by $l' \ge m > l$, $[T]_G$ is the $\langle G_{x_0,x_1,\ldots,x_{l'}}, g\rangle$-orbit containing the vertex $x_0$ from $h([T']_G)$. Hence $[T]_G \subseteq h([T']_G)$. At the same time, $[T']_G$ is the $\langle G_{x'_0,x'_1,\ldots,x'_{l'}}, g'\rangle$-orbit containing $x_0$. Since $G_{x'_0,x'_1,\ldots,x'_{l'}} \le G_{x_0,x_1,\ldots,x_m}$ and $g'g^{-1} \in G_{x_1,\ldots,x_m}$, it follows that $[T']_G$ is contained in the $\langle G_{x_1,\ldots,x_m}, g\rangle$-orbit containing $x_0$. But, as $m > l$, the $\langle G_{x_1,\ldots,x_m}, g\rangle$-orbit containing $x_0$ is $[T]_G$. Thus we have $[T']_G \subseteq [T]_G \subseteq h([T']_G)$. Since the subgraph of $\Gamma$ generated by $[T']_G$ and the subgraph of $\Gamma$ generated by $h([T']_G)$ are isomorphic connected (see Theorem 4.1(1)) locally finite graphs admitting vertex-transitive groups of automorphisms, it follows that $[T']_G = [T]_G = h([T']_G)$. Since $T$ is infinite, it follows that $T'$ is also infinite. Moreover, by the assertion (2), either $T$ and $T'$ or $T^{-1}$ and $T'$ are $G$-equivalent. If $T'$ and $T^{-1}$ are $G$-equivalent, then $\langle g^{-1}g'\rangle$-orbits on $[T']_G = [T]_G$ are infinite, contradicting $g^{-1}g'(x_0) = x_0$. Thus $T'$ and $T$ are $G$-equivalent. The claim is established.

There exists a positive integer $n_m$ such that $x_i = x'_{n_m,i} = x''_{n_m,i}$ for all $0 \le i \le m$. By the claim, $T'_{n_m}$ and $T$ are $G$-equivalent and $T''_{n_m}$ and $T$ are $G$-equivalent. Hence $T'_{n_m}$ and $T''_{n_m}$ are $G$-equivalent, contradicting the choice of $T'_{n_m}$ and $T''_{n_m}$.

(4) For any $G$-track $T$ of $\Gamma$, the stabilizer in $G$ of $[T]_G$ acts transitively on $[T]_G$. Thus the $G$-envelope of any $G$-track of $\Gamma$ containing $x$ is of the form $[T]_G$ where $T$ is a $G$-track passing through $x$, and the assertion (4) follows from (3).

(5) The assertion (5) follows from the assertion (3).

(6) Since the $G$-envelope of any $G$-track of $\Gamma$ containing $x$ is of the form $[T]_G$ where $T$ is a $G$-track passing through $x$, the assertion (6) follows from (5).

(7) Let $g$ be an element in $G$ such that $g^i(x_0) = x_i$ for all $i \in \mathbf{Z}$. Then $x_0 \in g^{-i}([T]_G) = [g^{-i}(T)]_G$ for all $0 \le i \le t_G$. Hence, by (6), there exist $0 \le i' < i'' \le t_G$ such that $g^{-i'}([T]_G) = g^{-i''}([T]_G)$. Thus the element $g^{i''-i'}$ stabilizes $[T]_G$. Since, for each $i \in \mathbf{Z}$, there exist $j \in \mathbf{Z}$ and $0 \le m \le t_G$ such that $g^i(x_0) = g^{(i''-i')j}(x_m)$, the result follows. $\square$

**Remark 4.6.** Let $n$ be an arbitrary positive integer. Then, for $\Gamma$ and $G$ satisfying the hypothesis of Corollary 4.5, the graph $\Gamma^n$ and the group $G$ regarded as a group of automorphisms of $\Gamma^n$ satisfy the hypothesis of Corollary 4.5 as well. Since each non-trivial generalized $g$-track $(\dots, x_{-1}, x_0, x_1, \dots)$ of $\Gamma$, $g \in G$, with $d_\Gamma(x_0, x_1) \le n$ is a $g$-track of $\Gamma^n$, analogies of assertions (3)–(7) of Corollary 4.5 hold also for such generalized $G$-tracks of $\Gamma$.

## 5. The modified track method. First applications

The modified track method arguments in the local approach to the Main Conjecture considerably depend on the group $R = G_x^{\Gamma(x)}$. Only some conceptual framework of the arguments is invariant. In this connection, in the present paper we prefer to demonstrate the method by means of concrete applications instead of trying to give a detailed description of the method in a general and abstract form. Nevertheless it seems that a few preliminary remarks and a previous outline of the method framework are appropriate. It should be mentioned that the following outline concerns the modified track method in a simple (or rather in an "ideal") form. It should be also mentioned that the modified track method in the local approach to the Main Conjecture (just like the track method for the FVSRP) is better adapted to transitive or even primitive groups $R$.

To prove that the Main Conjecture is valid for $\Gamma$ and $G$ with a fixed group $R = G_x^{\Gamma(x)}$, it is sufficient to consider the case when $G$ is a closed subgroup of $Aut(\Gamma)$. Of course we may also assume that $\Gamma$ is non-hyperbolic and $G_x$ is infinite. Now in the modified track method approach to the Main Conjecture the following two parts or steps can usually be recognized.

**Step 1.** The step consists in a search for a $g$-track $(\ldots, x_{-1}, x_0, x_1, \ldots)$ of $\Gamma$, where $g \in G$, such that $G^{[1]}_{x_1,\ldots,x_m} = G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ for some positive integer $m$.

In this section (see Theorem 5.1) we show that results of Section 4 can be used to realize Step 1 for many groups $R$.

**Step 2.** Let $(\ldots, x_{-1}, x_0, x_1, \ldots)$ be a $g$-track of $\Gamma$, where $g \in G$, such that $G^{[1]}_{x_1,\ldots,x_m} = G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ for some positive integer $m$ (see Step 1). We assume that $m$ is the smallest positive integer with this property for $(\ldots, x_{-1}, x_0, x_1, \ldots)$. Observe that $m > 1$ in the case where $R = G_x^{\Gamma(x)}$ is transitive. Since $G_{x_0}$ is infinite and $G^{[1]}_{x_1,\ldots,x_m} = G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ is a subgroup of finite index of $G_{x_0}$, the group $G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ is also infinite.

Put $N := N_G(G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots})$. Since $G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots} \neq 1$, the group $N$ acts intransitively on $V(\Gamma)$ and on $E(\Gamma)$. Since $g \in N$, it follows that $N_{x_i}$ acts intransitively on $\Gamma(x_i)$ for any integer $i$.

Put $H := \langle N_{x_0}, g\rangle \leq N$. By the choice of $m$, $G_{x_{i+1},\ldots,x_{i+m}} \leq H$ for any integer $i$. In addition $(G^{[1]}_{x_1,\ldots,x_{m-1}})^{\Gamma(x_0)}$ and $(G^{[1]}_{x_{-m+1},\ldots,x_{-1}})^{\Gamma(x_0)}$ are non-trivial subnormal subgroups of $H^{\Gamma(x_0)}_{x_0,x_1}$ and $H^{\Gamma(x_0)}_{x_0,x_{-1}}$ respectively (see (2.1)). Let $\Delta$ be the subgraph of $\Gamma$ generated by the $H$-orbit containing $x_0$. Then the graph $\Delta$ is connected, $x_i \in V(\Delta)$ for all $i \in \mathbf{Z}$, and $\tilde{H} := H^{V(\Delta)}$ is a vertex-transitive group of automorphisms of $\Delta$. In addition, $\tilde{H}_{x_0}$ is finite, since $H^{[1]}_{x_1,\ldots,x_m} = G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ acts trivially on $V(\Delta)$. Now the aim is to get a contradiction analyzing the action of $\tilde{H}$ on $\Delta$. Referring to subsequent examples for concrete realizations of Step 2, we only remark the following. Usually there are not so many possibilities for the group $\tilde{H}^{\Delta(x_0)}_{x_0}$ (this group is, in particular, a rather special section of $R$). If the FVSRP is solved affirmatively for each of them, we get a list of possibilities for $\tilde{H}_{x_0}$. (However it is more realistic to get a good upper bound for $m$.) Now to get a contradiction it is sufficient to eliminate these possibilities using, for example, the assumption that $G_{x_0}$ is infinite. (Frequently the assumption that $G_{x_0}$ is infinite implies a lower bound for $m$ contradicting the above upper bound.)

Note that frequently only some special $G$-track realizing Step 1 is also appropriate for Step 2. After all it looks like a surprise that such a track exists for many groups $R$.

Comparing the track method approach to the FVSRP (see the Appendix in the second part of the present paper) with the modified track method approach to the Main Conjecture, observe that certain arguments in Step 1 of the former and in Step 2 of the latter are similar, while Step 2 of the former and Step 1 of the latter are specific for each of these methods. (Step 1 of the latter is trivial for the former.)

Although realizations of Steps 1 and 2 are considerably dependent on the groups $R = G_x^{\Gamma(x)}$, results of Section 4 make it possible to prove the following Theorem 5.1 realizing Step 1 under fairly general assumptions on $R$.

**Theorem 5.1.** *Let $\Gamma$ be a connected locally finite non-hyperbolic graph, $G$ a vertex-transitive group of automorphisms of $\Gamma$, and $T = (\ldots, x_{-1}, x_0, x_1, \ldots)$ a $G$-track of the graph $\Gamma$. Suppose that $G^{[1]}_{x_1,\ldots,x_m} \neq G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ for any positive integer $m$. Then for each $\varepsilon \in \{-1, 1\}$, there exist $Z_\varepsilon \subseteq \Gamma(x_0)$ and $1 \neq A_\varepsilon \trianglelefteq\trianglelefteq G^{\Gamma(x_0)}_{x_0,x_\varepsilon}$ such that the following assertions hold.*

(a) *For all sufficiently large positive integers n, the* $G_{x_0,x_1,\ldots,x_n}$*-orbit containing* $x_{-1}$ *is contained in* $Z_{-1}$ *and the* $G_{x_{-n},\ldots,x_{-1},x_0}$*-orbit containing* $x_1$ *is contained in* $Z_1$*.*
(b) $|Z_{-1}| = |Z_1|$ *and* $Z_{-1} \cap Z_1 = \emptyset$*.*
(c) *For each* $\varepsilon \in \{-1, 1\}$*,* $A_\varepsilon$ *stabilizes pointwise the set* $Z_\varepsilon$ *and stabilizes (globally) the set* $Z_{-\varepsilon}$*.*

*If, further, G is a closed subgroup of* $Aut(\Gamma)$ *and* $V(\Gamma) \neq [T]_G$*, then we can choose* $Z_{-1}$ *and* $Z_1$ *with the additional property* $Z_{-1} \cup Z_1 \neq \Gamma(x_0)$*.*

**Proof.** Replacing, in case of need, $G$ by the closure of $G$ in $Aut(\Gamma)$, we assume without loss of generality that $G$ is a closed subgroup of $Aut(\Gamma)$.

Let $g$ be an element of $G$ such that $T$ is a $g$-track. Recall that $H_{T,G} := \langle G_{x_0,x_1,\ldots}, g\rangle$, and $K_{T,G}$ is the normal closure of $G_{x_0,x_1,\ldots}$ in $H_{T,G}$. For each integer $i$, let $X_i = K_{T,G}(x_i)$ be the $K_{T,G}$-orbit containing the vertex $x_i$. (Thus $[T]_G = \cup_{i\in\mathbf{Z}} X_i$.) By Theorem 4.1(3), the sets $X_i$, $i \in \mathbf{Z}$, are finite. In addition, by Theorem 4.1(2), $X_i \cap X_j = \emptyset$ for any integers $i \neq j$. According to Theorem 4.1(4), there exists a positive integer $l$ such that, for any integer $i$, the group $G_{x_i,x_{i+1},\ldots,x_{i+l}}$ stabilizes the set $X_k$ for each integer $k$.

Put $Z_{-1} := X_{-1} \cap \Gamma(x_0)$ and $Z_1 := X_1 \cap \Gamma(x_0)$. Observe that $V(\Gamma) = [T]_G$ in the case where $Z_{-1} \cup Z_1 = \Gamma(x_0)$.

For each $\varepsilon \in \{-1, 1\}$, we have $x_\varepsilon \in Z_\varepsilon$. Moreover, for any integer $n \geq l$, the $G_{x_0,x_1,\ldots,x_n}$-orbit containing $x_{-1}$ is contained in $Z_{-1}$ and the $G_{x_{-n},\ldots,x_{-1},x_0}$-orbit containing $x_1$ is contained in $Z_1$. Hence, for the sets $Z_{-1}$ and $Z_1$, (a) holds.

Next, since $X_{-1} \cap X_1 = \emptyset$, $Z_{-1} \cap Z_1 = \emptyset$. In addition, since the sets $X_{-1}$, $X_0$ and $X_1$ are $K_{T,G}$-orbits, $|Z_{-1}||X_0|$ is the number of edges of $\Gamma$ which are incident as with a vertex from $X_{-1}$ and with a vertex from $X_0$, and $|Z_1||X_0|$ is the number of edges of $\Gamma$ which are incident as with a vertex from $X_0$ and with a vertex from $X_1$. Since $g(X_{-1}) = X_0$ and $g(X_0) = X_1$, it follows that $|Z_{-1}| = |Z_1|$. Thus, for the sets $Z_{-1}$ and $Z_{-1}$, (b) holds.

For any integers $i \leq j$, put

$$N_{i,j} := \cap_{z\in X_i\cup\cdots\cup X_j} G_z^{[1]}.$$

In addition, put

$$N := \cap_{z\in\cdots\cup X_{-1}\cup X_0\cup X_1\cup\cdots} G_z^{[1]} = \cap_{z\in[T]_G} G_z^{[1]}.$$

It follows from [7, Theorem 2] (see (R1) in Section 2) that

$$|N_{i,j} : N_{i-1,j}| = |N_{i,j} : gN_{i-1,j}g^{-1}| = |N_{i,j} : N_{i,j+1}|$$

for any integers $i \leq j$. As a result, if $N_{i,j} = N_{i-1,j}$ or $N_{i,j} = N_{i,j+1}$ for some integers $i \leq j$, then $N_{i,j} = N$. But the equation $N_{i,j} = N$ is certainly impossible, since $N_{i,j}$ (being the stabilizer in $G_{x_i}$ of a finite set of vertices) is a subgroup of finite index of the group $G_{x_i}$ while $N \leq G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ where $G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ is a subgroup of infinite index of the group $G_{x_i}$ by the hypothesis of the theorem. Thus $N_{i-1,j} \neq N_{i,j} \neq N_{i,j+1}$ for any integers $i \leq j$. Since the group $K_{T,G}$ stabilizes the set $X_k$ for each $k \in \mathbf{Z}$ and acts transitively on the set $X_0$, it follows that

$$N_{1,j}^{\Gamma(x_0)} \neq 1, \qquad N_{-j,-1}^{\Gamma(x_0)} \neq 1 \tag{5.1}$$

for any integer $j \geq 1$.

Fix a positive integer $j \geq l-1$. By the choice of $l$, the group $G^{[1]}_{x_1,\ldots,x_j}$ stabilizes each set $X_k$, $1 \leq k \leq j$. Thus $N_{1,j} \trianglelefteq G^{[1]}_{x_1,\ldots,x_j}$. Analogously, since the group $G^{[1]}_{x_{-j},\ldots,x_{-1}}$ stabilizes each set $X_k$, $-j \leq k \leq -1$, we have $N_{-j,-1} \trianglelefteq G^{[1]}_{x_{-j},\ldots,x_{-1}}$. Since $G^{[1]}_{x_1,\ldots,x_j} \trianglelefteq\trianglelefteq G_{x_0,x_1}$ and $G^{[1]}_{x_{-j},\ldots,x_{-1}} \trianglelefteq\trianglelefteq G_{x_{-1},x_0}$, it follows that $N_{1,j} \trianglelefteq\trianglelefteq G_{x_0,x_1}$ and $N_{-j,-1} \trianglelefteq\trianglelefteq G_{x_{-1},x_0}$. Put $A_1 := N^{\Gamma(x_0)}_{1,j}$ and $A_{-1} := N^{\Gamma(x_0)}_{-j,-1}$. Then $A_\varepsilon \trianglelefteq\trianglelefteq G^{\Gamma(x_0)}_{x_0,x_\varepsilon}$ for each $\varepsilon \in \{-1, 1\}$. In addition, by (5.1), $A_\varepsilon \neq 1$ for each $\varepsilon \in \{-1, 1\}$. Since the group $N_{1,j}$ stabilizes pointwise the set $X_1$, and the group $N_{-j,-1}$ stabilizes pointwise the set $X_{-1}$, the group $A_\varepsilon$ stabilizes pointwise the set $Z_\varepsilon$ for each $\varepsilon \in \{-1, 1\}$. Finally, since the group $N_{1,j} \leq G_{x_0,x_1,\ldots,x_l}$ stabilizes the set $X_{-1}$, and the group $N_{-j,-1} \leq G_{x_{-l},\ldots,x_{-1},x_0}$ stabilizes the set $X_1$, the group $A_\varepsilon$ stabilizes the set $Z_{-\varepsilon}$ for each $\varepsilon \in \{-1, 1\}$. Thus for such $A_{-1}$, $A_1$, and for $Z_{-1}$, $Z_1$ defined above, (c) holds.

The proof is complete. □

**Corollary 5.2.** *Let $\Gamma$ be a connected locally finite non-hyperbolic graph, $G$ a vertex-transitive group of automorphisms of $\Gamma$, and $(\ldots, x_{-1}, x_0, x_1, \ldots)$ a $G$-track of the graph $\Gamma$. Suppose that, for any non-trivial subnormal subgroup $A$ of the group $G^{\Gamma(x_0)}_{x_0,x_1}$, the length of the $A$-orbit containing $x_{-1}$ is greater than the number of fixed vertices of $A$ (on $\Gamma(x_0)$). Then there exists a positive integer $m$ such that $G^{[1]}_{x_1,\ldots,x_m} = G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$.*

Theorem 5.1 combined with some of arguments presented in the beginning of this section gives the following.

**Corollary 5.3.** *Let $\Gamma$ be a connected locally finite non-hyperbolic graph, $G$ a vertex-transitive closed subgroup of $Aut(\Gamma)$, and $T = (\ldots, x_{-1}, x_0, x_1, \ldots)$ a $G$-track of the graph $\Gamma$. Suppose that $G^{\Gamma(x_0)}_{x_0,x_1} \neq 1$ (or, equivalently, that $G^{\Gamma(x_0)}_{x_{-1},x_0} \neq 1$; see* (R1) *and* (iv) *in Section 2). In addition, suppose that, for any non-trivial subnormal subgroup $S_1$ of the group $G^{\Gamma(x_0)}_{x_0,x_1}$ and for any non-trivial subnormal subgroup $S_{-1}$ of the group $G^{\Gamma(x_0)}_{x_{-1},x_0}$, the equation*

$$\langle S_{-1}, S_1\rangle(x_{-1}) \cup \langle S_{-1}, S_1\rangle(x_1) = \Gamma(x_0)$$

*holds. Then $G_{x_0}$ is finite or $V(\Gamma) = [T]_G$. (In particular, for $\Gamma$ and $G$, the possibility* (1) *from the Main Conjecture holds; see Remark* 4.2.*)*

**Proof.** Assume that $G_{x_0}$ is infinite and $V(\Gamma) \neq [T]_G$. If $G^{[1]}_{x_1,\ldots,x_m} \neq G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ for any positive integer $m$, then, by Theorem 5.1, there exist $1 \neq A_1 \trianglelefteq\trianglelefteq G^{\Gamma(x_0)}_{x_0,x_1}$, $1 \neq A_{-1} \trianglelefteq\trianglelefteq G^{\Gamma(x_0)}_{x_0,x_{-1}}$ and $\langle A_{-1}, A_1\rangle$-invariant subsets $Z_{-1}$ and $Z_1$ of $\Gamma(x_0)$ such that $x_{-1} \in Z_{-1}$, $x_1 \in Z_1$ and $Z_{-1} \cup Z_1 \neq \Gamma(x_0)$. Setting $S_1 = A_1$ and $S_{-1} = A_{-1}$, we have

$$\langle S_{-1}, S_1\rangle(x_{-1}) \cup \langle S_{-1}, S_1\rangle(x_1) \subseteq Z_{-1} \cup Z_1 \neq \Gamma(x_0)$$

contrary to the hypothesis of the corollary. Thus $G^{[1]}_{x_1,\ldots,x_m} = G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ for some positive integer $m$. Take $m$ to be the smallest positive integer with this property. Since $G_x$ is infinite, $G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots} \neq 1$. Put $N := N_G(G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots})$. Then $N$ is intransitive on $V(\Gamma)$. On the other hand, by the choice of $m$, we have $G^{[1]}_{x_{-m+1},\ldots,x_{-1}} \leq N$ and $G^{[1]}_{x_1,\ldots,x_{m-1}} \leq N$ in the case $m > 1$, while $G_{x_{-1},x_0} \leq N$ and $G_{x_0,x_1} \leq N$ in the case $m = 1$. In addition, by the choice of $m$, we have $(G^{[1]}_{x_{-m+1},\dots,x_{-1}})^{\Gamma(x_0)} \neq 1$ and $(G^{[1]}_{x_1,\dots,x_{m-1}})^{\Gamma(x_0)} \neq 1$ in the case $m > 1$ (see (2.1)). By the hypothesis of the corollary, setting $S_1 = (G^{[1]}_{x_1,\dots,x_{m-1}})^{\Gamma(x_0)} \leq N_{x_0}^{\Gamma(x_0)}$ and $S_{-1} = (G^{[1]}_{x_{-m+1},\dots,x_{-1}})^{\Gamma(x_0)} \leq N_{x_0}^{\Gamma(x_0)}$ in the case $m > 1$, and setting $S_1 = G_{x_0,x_1}^{\Gamma(x_0)} \leq N_{x_0}^{\Gamma(x_0)}$ and $S_{-1} = G_{x_{-1},x_0}^{\Gamma(x_0)} \leq N_{x_0}^{\Gamma(x_0)}$ in the case $m = 1$, we have

$$\Gamma(x_0) = \langle S_{-1}, S_1\rangle(x_{-1}) \cup \langle S_{-1}, S_1\rangle(x_1) \subseteq N_{x_0}(x_{-1}) \cup N_{x_0}(x_1).$$

Since $g \in N$, where $g$ is an element of $G$ such that $T$ is a $g$-track it follows that $N$ is vertex-transitive, a contradiction. $\square$

**Remark 5.4.** It is easy to see that, in Corollary 5.3, the supposition $G_{x_0,x_1}^{\Gamma(x_0)} \neq 1$ can be omitted in the case where $G_{x_0}^{\Gamma(x_0)}$ is transitive.

It follows from Corollary 5.3 that the Main Conjecture is valid, for example, in the case when $R = G_x^{\Gamma(x)}$ contains a regular normal subgroup of prime order.

On the whole now we are in a position to apply the modified track method arguments to concrete groups $R$ within the local approach to the Main Conjecture.

**Example 5.5.** As a very simple application of the modified track method (in the form of Corollary 5.3), we show that the Main Conjecture is valid in the case when the group $R = G_x^{\Gamma(x)}$ is the dihedral group $D_{2n}$, $n \geq 3$, acting in the natural way on $n$ points. (It is well known that, under the assumptions of the FVSRP with such $R$ and $n = 4$, the order of the stabilizer of a vertex of $\Gamma$ in $G$ can be arbitrarily large.)

Without loss we assume that $G$ is a closed subgroup of $Aut(\Gamma)$. We also assume that the graph $\Gamma$ is non-hyperbolic.

Let $y \in \Gamma(x)$. Then $G_x^{\Gamma(x)}$ is a semidirect product of a regular normal cyclic subgroup $\langle a\rangle$ of order $n$ by the subgroup $G_{x,y}^{\Gamma(x)} = \langle b\rangle$ of order 2, where $bab = a^{-1}$. Let $z = a(y)$. Then $G_{x,z}^{\Gamma(x)} = \langle aba^{-1}\rangle$. There exists $g \in G$ such that $g(z) = x$ and $g(x) = y$. For each $i \in \mathbf{Z}$, put $x_i := g^i(x)$. Then $T = (\dots, x_{-1}, x_0, x_1, \dots)$ is a $g$-track of $\Gamma$. The unique non-trivial subgroup of $G_{x_0,x_1}^{\Gamma(x_0)}$ and the unique non-trivial subgroup of $G_{x_{-1},x_0}^{\Gamma(x_0)}$ generate in $G_{x_0}^{\Gamma(x_0)}$ the subgroup $\langle a^2, b\rangle$. Since

$$\langle a^2, b\rangle(x_{-1}) \cup \langle a^2, b\rangle(x_1) = \langle a^2, b\rangle(a(x_1)) \cup \langle a^2, b\rangle(x_1) = \Gamma(x_0),$$

it follows from Corollary 5.3 that $V(\Gamma) = [T]_G$. In particular, for $\Gamma$ and $G$, the possibility (1) of the Main Conjecture holds (see Remark 4.2).

**Example 5.6.** As another rather simple application of the modified track method, we show that the Main Conjecture is valid in the case when the group $R = G_x^{\Gamma(x)}$ contains a normal subgroup which is one of the following groups:

(a) $PSU_3(q)$, $q > 2$ a power of a prime $p$, acting in the natural way on $q^3 + 1$ points;
(b) $Sz(q)$, $q = 2^{2k+1} > 2$, acting in the natural way on $q^2 + 1$ points;
(c) ${}^2G_2(q)$, $q = 3^{2k+1} \geq 3$, acting in the natural way on $q^3 + 1$ points.

Without loss we may assume that $G$ is a closed subgroup of $Aut(\Gamma)$. We also assume that $G_x$ is an infinite group and $\Gamma$ is non-hyperbolic. Put $p = 2$ if (b) holds, and $p = 3$ if (c) holds.

Let $y \in \Gamma(x)$. It is easy to see that any non-trivial subnormal subgroup of $G_{x,y}^{\Gamma(x)}$ has a non-trivial intersection with the group $O_p(G_{x,y}^{\Gamma(x)})$ which acts regularly on $\Gamma(x) \setminus \{y\}$. Hence, by Corollary 5.2, if $(\ldots, x_{-1}, x_0, x_1, \ldots)$ is an arbitrary $G$-track of the graph $\Gamma$, then there exists a positive integer $m$ such that $G_{x_1,\ldots,x_m}^{[1]} = G_{\ldots,x_{-1},x_0,x_1,\ldots}^{[1]}$.

Let $z \in \Gamma(x) \setminus \{y\}$. Then $(G_{x,y}^{[1]})^{\Gamma(z)}$ is a subnormal subgroup of $G_{x,z}^{\Gamma(z)}$. We claim that $(G_{x,y}^{[1]})^{\Gamma(z)}$ is a $p$-group (and hence $(G_{x,y}^{[1]})^{\Gamma(z)} \leq O_p(G_{x,z}^{\Gamma(z)})$). Suppose not. Since $\Gamma$ is not a regular tree, there exists a maximal integer $s > 1$ such that, for some $s$-arc $(y_0, y_1, \ldots, y_s)$ of $\Gamma$, $G_{y_0,y_1,\ldots,y_s}^{\Gamma(y_0)} \geq O_p(G_{y_0,y_1}^{\Gamma(y_0)})$. Observe that, for such $s$, $G$ is $(s+1)$-arc-transitive. In addition, $G_{y_0,y_1,\ldots,y_s}^{\Gamma(y_0)}$ contains a Hall $p'$-subgroup of $G_{y_0,y_1}^{\Gamma(y_0)}$ (since $|G_{y_0,y_1} : G_{y_0,y_1,\ldots,y_s}|$ is a power of $p$). At the same time, since $(G_{x,y}^{[1]})^{\Gamma(z)}$ is not a $p$-group, it follows from Proposition 2.5 that $G_{y_s}^{[s]}/G_{y_s}^{[s+1]}$ also is not a $p$-group. Moreover, by $s$-arc-transitivity of $G$, the subnormal subgroup $(G_{y_s}^{[s]})^{\Gamma(y_0)}$ of $G_{y_0,y_1}^{\Gamma(y_0)}$ is not a $p$-group. Thus, since in all cases (a), (b) and (c) a Hall $p'$-subgroup of $G_{y_0,y_1}^{\Gamma(y_0)}$ acts irreducibly on $O_p(G_{y_0,y_1}^{\Gamma(y_0)})/\Phi(O_p(G_{y_0,y_1}^{\Gamma(y_0)}))$ and the group $G_{y_0,y_1}^{\Gamma(y_0)}/O_p(G_{y_0,y_1}^{\Gamma(y_0)})$ acts faithfully on $O_p(G_{y_0,y_1}^{\Gamma(y_0)})/\Phi(O_p(G_{y_0,y_1}^{\Gamma(y_0)}))$, we have

$$O_p(G_{y_0,y_1}^{\Gamma(y_0)}) \leq [(G_{y_s}^{[s]})^{\Gamma(y_0)}, G_{y_0,y_1,\ldots,y_s}^{\Gamma(y_0)}].$$

But

$$[G_{y_s}^{[s]}, G_{y_0,y_1,\ldots,y_s}] \leq G_{y_s}^{[s]} \leq G_{y_0,y_1,\ldots,y_s,y_{s+1}}$$

for any $y_{s+1} \in \Gamma(y_s) \setminus \{y_{s-1}\}$, and the choice of $s$ gives a contradiction. The claim is established.

Next, for any positive integer $n$, the group $G_{x,y}/G_{x,y}^{[n]}$ is solvable. Thus there exists a subgroup $D$ of the group $G_{x,y}$ such that, for any positive integer $n$, $(DG_{x,y}^{[n]})/G_{x,y}^{[n]}$ is a Hall $p'$-subgroup of $G_{x,y}/G_{x,y}^{[n]}$. (Note that, by the above, $D$ acts faithfully on $\Gamma(x) \cup \Gamma(y)$ and, in particular, is a finite $p'$-group.) Moreover, the set of such subgroups $G_{x,y}$ is a conjugacy class of subgroups of $G_{x,y}$, and any such subgroup of $G_{x,y}$ stabilizes a vertex in $\Gamma(x) \setminus \{y\}$. Let $D$ be such a subgroup of $G_{x,y}$ stabilizing the vertex $z \in \Gamma(x) \setminus \{y\}$. Then $D$ is also a subgroup of the group $G_{x,z}$ such that, for any positive integer $n$, $(DG_{x,z}^{[n]})/G_{x,z}^{[n]}$ is a Hall $p'$-subgroup of $G_{x,z}/G_{x,z}^{[n]}$. There is $g \in G$ such that $g(z) = x$ and $g(x) = y$. Multiplying (in case of need) $g$ by an appropriate element of $G_{x,y}$, we can assume without loss of generality that $g$ normalizes $D$.

For each $i \in \mathbf{Z}$, put $x_i := g^i(x)$. Then $(\ldots, x_{-1}, x_0, x_1, \ldots)$ is a $g$-track of $\Gamma$. Henceforth, let $m$ be the smallest positive integer with the property $G_{x_1,\ldots,x_m}^{[1]} = G_{\ldots,x_{-1},x_0,x_1,\ldots}^{[1]}$ (such an $m$ exists; see the beginning of the proof). Observe that $m > 3$, since otherwise $(G_{x_1,x_2,x_3}^{[1]})^{\Gamma(x_0)} = 1$ and, by 2-arc-transitivity of $G$, $G_x^{[2]} = 1$ contrary to the assumption that $G_x$ is infinite. Observe also that the group $G_{\ldots,x_{-1},x_0,x_1,\ldots}^{[1]}$ is infinite, since $G_{x_1,\ldots,x_m}^{[1]}$ is a subgroup of finite index of the infinite group $G_{x_0}$.

Put $N := N_G(G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots})$. Then $N$ is intransitive on $V(\Gamma)$. Since $g \in N$, it follows that $N_{x_0}^{\Gamma(x_0)}$ is intransitive. At the same time, we have $D \le N$, $G^{[1]}_{x_{-m+1},\ldots,x_{-1}} \le N$ and $G^{[1]}_{x_1,\ldots,x_{m-1}} \le N$. Since $1 \ne (G^{[1]}_{x_{-m+1},\ldots,x_{-1}})^{\Gamma(x_0)} \trianglelefteq\trianglelefteq G^{\Gamma(x_0)}_{x_{-1},x_0}$ and $1 \ne (G^{[1]}_{x_1,\ldots,x_{m-1}})^{\Gamma(x_0)} \trianglelefteq\trianglelefteq G^{\Gamma(x_0)}_{x_0,x_1}$ (see (2.1)), we conclude using well-known properties of groups from (a), (b) and (c) that the following assertions, concerning cases (a), (b) and (c) respectively, hold.

If (a) holds, then $N^{\Gamma(x_0)}_{x_0,x_\varepsilon} \cap O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}) = \Phi(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon})) = Z(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$ for each $\varepsilon \in \{-1,1\}$. (This is easily seen since, in case (a), the group $D^{\Gamma(x_0)}$ acts irreducibly on $\Phi(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$ and on $O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon})/\Phi(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$ for each $\varepsilon \in \{-1,1\}$.) Moreover, denoting the $N_{x_0}$-orbit containing $x_{-1}$ by $Z$, we have $x_1 \in Z$, $|Z| = q+1$ and the group $N^Z_{x_0}$ contains a normal subgroup which is $PSL_2(q)$ in the usual doubly transitive representation.

Case (b) is impossible. (In fact, in case (b), the group $D^{\Gamma(x_0)}$ acts irreducibly on $\Phi(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$ and on $O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon})/\Phi(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$ for each $\varepsilon \in \{-1,1\}$. Thus $\langle \Phi(O_p(G^{\Gamma(x_0)}_{x_0,x_{-1}})), \Phi(O_p(G^{\Gamma(x_0)}_{x_0,x_1}))\rangle \le N^{\Gamma(x_0)}_{x_0}$ contrary to the intransitivity of $N^{\Gamma(x_0)}_{x_0}$.)

Finally, if (c) holds, then, for each $\varepsilon \in \{-1,1\}$, $N^{\Gamma(x_0)}_{x_0,x_\varepsilon} \cap O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}) = A_\varepsilon$ is the only $D^{\Gamma(x_0)}$-invariant elementary abelian subgroup of order $q$ of $O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon})$ different from $Z(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$. (In fact, in case (c) with $q > 3$, the group $D^{\Gamma(x_0)}$ acts irreducibly on $Z(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$, on $\Phi(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))/Z(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$ and on $O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon})/\Phi(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$ for each $\varepsilon \in \{-1,1\}$. In addition, in case (c), $\langle Z(O_p(G^{\Gamma(x_0)}_{x_0,x_{-1}})), Z(O_p(G^{\Gamma(x_0)}_{x_0,x_{-1}}))\rangle$ is transitive, and, for each $\varepsilon \in \{-1,1\}$, there is a unique $D^{\Gamma(x_0)}$-invariant elementary abelian subgroup of order $q$ of $O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon})$ different from $Z(O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon}))$. This subgroup is the centralizer in $O_p(G^{\Gamma(x_0)}_{x_0,x_\varepsilon})$ of the involution from $D^{\Gamma(x_0)}$.) Moreover, denoting the $N_{x_0}$-orbit containing $x_{-1}$ by $Z$, we have $x_1 \in Z$ and either $|Z| = q+1$ and the group $N^Z_{x_0}$ contains a normal subgroup which is $PSL_2(q)$ in the usual doubly transitive representation (in this case $Z$ is the set of fixed vertices of the involution in $G^{\Gamma(x_0)}_{x_{-1},x_0,x_1}$), or $q = 3$, $|Z| = 7$ and $N^Z_{x_0}$ is the doubly transitive Frobenius group of order 42.

In the remaining subcases of cases (a) and (c), let $H := \langle N_{x_0}, g\rangle$ and let $\Delta$ be the subgraph of $\Gamma$ generated by the $H$-orbit containing $x_0$. To complete the proof, we consider the action of the group $\tilde{H} := H^{V(\Delta)}_{\{V(\Delta)\}} \le Aut(\Delta)$. Observe that, by the above, the graph $\Delta$ is connected, $x_i \in V(\Delta)$ for all $i \in \mathbf{Z}$, $\tilde{H}$ is vertex-transitive, $\tilde{H}_{x_0}$ is finite (since $H^{[1]}_{x_1,\ldots,x_m} = G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$ acts trivially on $V(\Delta)$), $\Delta(x_0) = Z$ and either $\tilde{H}^{\Delta(x_0)}_{x_0}$ contains a normal subgroup which is $PSL_2(q)$ in the usual doubly transitive representation (and $|\Delta(x_0)| = q+1$) or $\tilde{H}^{\Delta(x_0)}_{x_0}$ is the doubly transitive Frobenius group of order 42 (and $|\Delta(x_0)| = 7$, $q = 3$).

Since $G^{[1]}_{x_1,x_2,\ldots,x_{m-1}} \le H$ where $m > 3$ and $(G^{[1]}_{x_1,x_2,\ldots,x_{m-1}})^{\Delta(x_0)} \ne 1$, it follows that $\tilde{H}^{[1]}_{x_1,x_2,x_3} \ne 1$. Therefore, $\tilde{H}^{\Delta(x_0)}_{x_0}$ cannot be the doubly transitive Frobenius group of order 42 (see [15]). Thus $\tilde{H}^{\Delta(x_0)}_{x_0}$ has a normal subgroup which is $PSL_2(q)$ in the usual doubly transitive representation. By [14], it follows that

$$\tilde{H}^{[1]}_{x_1,x_2,x_3} = \tilde{H}^{[2]}_{x_2}. \tag{5.2}$$

Furthermore, since $\tilde{H}^{[1]}_{x_1,x_2,x_3} \neq 1$, it follows from [14] that $p \leq 3$ and, in the case $p = 3$, the group $\tilde{H}^{\Delta(x_0)}_{x_0,x_1}$ contains an involution. Since, by the above, $G^{[1]}_{x_1,x_2,\ldots,x_{m-1}} = H^{[1]}_{x_1,x_2,\ldots,x_{m-1}}$ acts transitively on $\Delta(x_0) \setminus \{x_1\}$ where $m > 3$, it also follows from [14] that $m \leq 4$ for $p = 2$ and $m \leq 6$ for $p = 3$.

Now we consider the remaining subcase of case (c). Since $p = 3$, it follows from the above that $\tilde{H}^{\Delta(x_0)}_{x_0,x_1}$ contains an involution. Therefore there exists a 2-element of the group $G^{\Gamma(x_0)}_{x_0,x_1}$ acting non-trivially on the set $\Delta(x_0)$ which, in the remaining subcase of case (c), is the set of fixed vertices of the involution in $G^{\Gamma(x_0)}_{x_{-1},x_0,x_1}$. This contradicts (c).

To complete the proof we consider the remaining subcase of the case (a). Now the $D^{\Gamma(x_3)}$-invariant subnormal subgroup $(G^{[1]}_{x_1,x_2})^{\Gamma(x_3)}$ of $O_p(G^{\Gamma(x_3)}_{x_2,x_3})$ is either $O_p(G^{\Gamma(x_3)}_{x_2,x_3})$ or $\Phi(O_p(G^{\Gamma(x_3)}_{x_2,x_3}))$ (see the beginning of the proof).

Assume first that

$$(G^{[1]}_{x_1,x_2})^{\Gamma(x_3)} = O_p(G^{\Gamma(x_3)}_{x_2,x_3}).$$

Then $G$ is 4-arc-transitive. Since $G^{[4]}_x \neq 1$, it follows that $(G^{[4]}_{x_4})^{\Gamma(x_0)} \neq 1$. But then $G^{[1]}_{x_1,x_2,\ldots,x_7} \not\leq G^{[1]}_{\ldots,x_{-1},x_0,x_1,\ldots}$, contradicting $m \leq 6$.

Finally assume that

$$(G^{[1]}_{x_1,x_2})^{\Gamma(x_3)} = \Phi(O_p(G^{\Gamma(x_3)}_{x_2,x_3})). \tag{5.3}$$

Since $(G^{[1]}_{x_{-m+1},\ldots,x_{-2},x_{-1}})^{\Gamma(x_0)} = \Phi(O_p(G^{\Gamma(x_0)}_{x_{-1},x_0}))$, it follows that

$$(G^{[1]}_{x_1,\ldots,x_{i-1}})^{\Gamma(x_i)} = \Phi(O_p(G^{\Gamma(x_i)}_{x_{i-1},x_i})) = (G^{[1]}_{x_{i-m+1},\ldots,x_{i-2},x_{i-1}})^{\Gamma(x_i)}$$

for all $3 \leq i \leq m$. Since $G^{[1]}_{x_{i-m+1},\ldots,x_{i-2},x_{i-1}} \leq H$ for all integers $i$, this gives

$$G^{[1]}_{x_1,x_2} \leq H. \tag{5.4}$$

Let $x'$ be an arbitrary vertex in $\Delta(x_2) \setminus \{x_1\}$ (i.e., an arbitrary vertex in the $\Phi(O_p(G^{\Gamma(x_2)}_{x_1,x_2}))$-orbit containing $x_3$). Then, by (5.3), $(G^{[1]}_{x_1,x_2})^{\Gamma(x')} = \Phi(O_p(G^{\Gamma(x')}_{x_2,x'}))$. Hence $(H^{[1]}_{x_1,x_2,x_3})^{\Gamma(x')} \leq \Phi(O_p(G^{\Gamma(x')}_{x_2,x'}))$. By (5.2), it follows that $(H^{[1]}_{x_1,x_2,x_3})^{\Gamma(x')} = 1$. Since, by (5.4), $G^{[1]}_{x_1,x_2,x_3} = H^{[1]}_{x_1,x_2,x_3}$, we conclude that $(G^{[1]}_{x_1,x_2,x_3})^{\Gamma(x')} = 1$.

Since $G$ is 2-arc-transitive, it follows that $(G^{[1]}_{x,z_1,z_2})^{\Gamma(z')} = 1$ for any $z_1 \in \Gamma(x)$ and $z_2 \in \Gamma(x) \setminus \{z_1\}$ and for any $z'$ in the $\Phi(O_p(G^{\Gamma(x)}_{z_1,x}))$-orbit containing $z_2$. Since we consider a subcase of case (a), it follows that there exist $z_1 \in \Gamma(x)$, $z_2 \in \Gamma(x) \setminus \{z_1\}$ and $z_3 \in \Gamma(x) \setminus \{z_1, z_2\}$ such that

$$G^{[1]}_{x,z_1,z_2,z_3} = G^{[2]}_x. \tag{5.5}$$

Let $Q := O_p(G_x/G^{[2]}_x) = O_p(G^{[1]}_x/G^{[2]}_x)$. By (5.3) and 2-arc-transitivity of $G$, we have $(G^{[1]}_{x,z_1})^{\Gamma(z_i)} = \Phi(O_p(G^{\Gamma(z_i)}_{x,zi}))$ for $i = 2, 3$. Hence, by (5.5), $G^{[1]}_{x,z_1}/G^{[2]}_x$ is a non-trivial $p$-subgroup of order $\leq q^2$ of the group $Q$. Since, by (a), any non-trivial normal subgroup of $G^{\Gamma(x)}_x$ is transitive, it follows that

$$C_{G_x/G^{[2]}_x}(Q) \leq G^{[1]}_x/G^{[2]}_x. \tag{5.6}$$

Furthermore, we conclude that $Q$ is the preimage of $O_p(G_x^{[1]}/G_{x,z_1}^{[1]})$ under the natural homomorphism $G_x^{[1]}/G_x^{[2]} \to G_x^{[1]}/G_{x,z_1}^{[1]}$, and hence $(G_x^{[1]}/G_x^{[2]})/Q$ is isomorphic to $(G_x^{[1]}/G_{x,z_1}^{[1]})/O_p(G_x^{[1]}/G_{x,z_1}^{[1]})$. Thus, $(G_x^{[1]}/G_x^{[2]})/Q$ is isomorphic to a subgroup of the group $G_{x,z_1}^{\Gamma(z_1)}/O_p(G_{x,z_1}^{\Gamma(z_1)})$ which, by (a), is isomorphic to a subgroup of $N_{P\Gamma U_3(q)}(P)/P$ where $P$ is a Sylow $p$-subgroup of $PSU_3(q) \trianglelefteq P\Gamma U_3(q)$. By (a) and (5.6), it follows that $(G_x/G_x^{[2]})/Q$ contains a subgroup $L$ which is isomorphic to $PSU_3(q)$ or $SU_3(q)$ and acts non-trivially on $Q$. Since $|Q| \leq q^5$, we get a non-trivial $\mathbf{F}_p$-module of $L$ of order $\leq q^5$. But any non-trivial $\mathbf{F}_p$-module of $PSU_3(q)$ or $SU_3(q)$ is of order $\geq q^6$ (see [2, (5.7)]), a contradiction.

**Remark 5.7.** If follows from arguments at the beginning of the consideration of Example 5.6 and, for example, from [3, Theorem 2.6] that, under assumptions of Example 5.6, if $(G_{x,y}^{[1]})^{\Gamma(z)}$ is not a $p$-group for $y \in \Gamma(x)$ and $z \in \Gamma(x) \setminus \{y\}$ (where $p = 2$ if (b) holds and $p = 3$ if (c) holds), then $\Gamma$ is a regular tree.

**Remark 5.8.** It follows from Remark 3.4 and Example 5.6 that, for $\Gamma$ and $G$ satisfying the hypothesis of the Main Conjecture and for $x \in V(\Gamma)$, if $\Gamma$ is non-hyperbolic, $G_x$ is infinite and $G_x^{\Gamma(x)}$ is a doubly transitive group with a simple non-abelian socle, then only case (a) of Example 3.7 (i.e., case 2 with $d > 2$ from Table 7.4 of [1]) is possible for $G_x^{\Gamma(x)}$. In Example 3.7, we started to consider this case as well. As was mentioned there, we will continue to consider this case in the second part of the paper.

Some other applications of the modified track method approach will be given in the second part of the present paper.

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